

Epicyclic oscillations of fluid bodies: II. Strong gravity

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Abstract

Fluids in external gravity may oscillate with frequencies characteristic of the epicyclic motions of test particles. We explicitly demonstrate that global oscillations of a slender, perfect fluid torus around a Kerr black hole admit incompressible vertical and radial epicyclic modes. Our results may be directly relevant to one of the most puzzling astrophysical phenomena—high (hundreds of hertz) frequency quasiperiodic oscillations (QPOs) detected in x-ray fluxes from several black hole sources. Such QPOs are pairs of stable frequencies in the 3/2 ratio. It seems that they originate a few gravitational radii away from the black hole and thus their observations have the potential to become an accurate probe of super-strong gravity.

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1. Introduction: QPOs and epicyclic frequencies

High frequency (hundreds of hertz), quasiperiodic variations in x-ray fluxes have been detected in several Galactic black hole sources by x-ray telescopes on spacecraft (e.g. van der Klis (2000, 2005) and McClintock and Remillard (2004)). They are called QPOs and have frequencies

corresponding to orbital frequencies a few gravitational radii away from black holes. Of special interest are the twin peak QPOs that come in pairs of two frequencies in the 3/2 ratio.

Kluźniak and Abramowicz (2000, 2001) and their collaborators developed a model of twin peak QPOs in terms of a nonlinear resonance in epicyclic oscillations of nearly geodesic motion of matter close to the black hole (see Abramowicz (2005) for information and references). In this paper, we extend the model by studying epicyclic oscillations of relativistic fluid tori. We prove that *in an axially symmetric stationary spacetime, an equilibrium perfect fluid torus with small cross-section always admits global epicyclic eigenmodes of oscillations*. In Newtonian theory, the same was proved recently by Blaes *et al* (2005). We follow closely their method, adapting it to general relativity. We start with a short reminder of how to construct an equilibrium model of the torus (after Abramowicz *et al* (1978)).

2. Equilibrium of the torus

The line element of a stationary, axially symmetric spacetime is given in the spherical coordinates (t, r, θ, ϕ) by

$$ds^2 = g_{tt} dt^2 + 2g_{t\phi} dt d\phi + g_{rr} dr^2 + g_{\theta\theta} d\theta^2 + g_{\phi\phi} d\phi^2. \quad (1)$$

We use the $-+++$ signature. The metric coefficients depend on r and θ , but not on t and ϕ , and for this reason it is convenient to introduce two Killing vectors $\eta^\alpha = \delta^\alpha_t$ and $\xi^\alpha = \delta^\alpha_\phi$. Note that, obviously, $\eta^\alpha \eta_\alpha = g_{tt}$, $\xi^\alpha \eta_\alpha = g_{t\phi}$, $\xi^\alpha \xi_\alpha = g_{\phi\phi}$. We assume in addition, reflection symmetry, i.e. that all odd θ -derivatives of the metric coefficients vanish in the symmetry plane $\theta = \pi/2$.

2.1. Circular orbits and circular geodesics

We start by briefly recalling some known results (see e.g. Wald (1984)) concerning circular orbits and circular geodesics. A circular orbit is given by

$$u^\alpha = A(\eta^\alpha + \Omega \xi^\alpha), \quad (2)$$

with Ω being the angular velocity, and $1/A^2 = -(\eta^\alpha \eta_\alpha + 2\Omega \eta^\alpha \xi_\alpha + \Omega^2 \xi^\alpha \xi_\alpha)$. The conserved energy and angular momentum are given by $\mathcal{E} = -\eta^\alpha u_\alpha$, $\mathcal{L} = \xi^\alpha u_\alpha$ and the conserved specific angular momentum by $\ell = \mathcal{L}/\mathcal{E}$. The acceleration at the circular orbit is

$$a_\alpha \equiv u^\beta \nabla_\beta u_\alpha = -\frac{1}{2} \left(\frac{\partial_\alpha g^{tt} - 2\ell \partial_\alpha g^{t\phi} + \ell^2 \partial_\alpha g^{\phi\phi}}{g^{tt} - 2\ell g^{t\phi} + \ell^2 g^{\phi\phi}} \right) = -\frac{1}{2\mathcal{U}} \left(\frac{\partial \mathcal{U}}{\partial x^\alpha} \right)_\ell, \quad (3)$$

where $\mathcal{U} = g^{tt} - 2\ell g^{t\phi} + \ell^2 g^{\phi\phi}$ is the effective potential.

The *geodesic circle* is defined by the condition $a_\alpha = 0$. Radial ω_r and vertical ω_θ epicyclic frequencies for circular geodesics around black holes were derived first by Kato and Fukue (1980); see also Silbergleit *et al* (2001). They may be expressed in terms of the second derivatives of the effective potential (Wald 1984, Abramowicz and Kluźniak 2002),

$$\omega_r^2 = \frac{\mathcal{E}^2}{2A^2 g_{rr}} \left(\frac{\partial^2 \mathcal{U}}{\partial r^2} \right)_\ell, \quad \omega_\theta^2 = \frac{\mathcal{E}^2}{2A^2 g_{\theta\theta}} \left(\frac{\partial^2 \mathcal{U}}{\partial \theta^2} \right)_\ell. \quad (4)$$

These are frequencies measured by an observer at infinity. Note that these expressions are invariant—independent of the choice of metric signature and coordinates.

2.2. Perfect fluid

The torus is made of perfect fluid with stress–energy tensor $T^\alpha{}_\beta = (e + p)u^\alpha u_\beta + p\delta^\alpha{}_\beta$, where e and p are the energy density and pressure of the flow, respectively. The energy density consists of rest mass density ρ and thermodynamic internal energy density U .

The dynamics is governed by the conservation law $\nabla_\alpha T^\alpha{}_\beta = 0$. Projection of this into the 3-space perpendicular to the 4-velocity, together with the 4-acceleration formula (3), gives

$$-\frac{\nabla_\alpha p}{e + p} - \frac{1}{2}\mathcal{E}_0^2 \nabla_\alpha \mathcal{U} = \frac{1}{2}(\mathcal{E}^2 - \mathcal{E}_0^2) \nabla_\alpha g^{tt} - (\mathcal{E}^2 \ell - \mathcal{E}_0^2 \ell_0) \nabla_\alpha g^{t\phi} + \frac{1}{2}(\mathcal{E}^2 \ell^2 - \mathcal{E}_0^2 \ell_0^2) \nabla_\alpha g^{\phi\phi}. \quad (5)$$

From (5), it follows that the pressure maximum $\nabla^\alpha p = 0$ is a geodesic circle $a^\alpha = 0$ and lies in the equatorial plane $\cos\theta = 0$. We shall call this circle the torus centre and indicate by subscript 0 quantities evaluated there; in particular \mathcal{E}_0, ℓ_0 are the energy and specific angular momentum at the torus centre, respectively.

2.3. Barotropic fluid

For a barotropic fluid $p = p(e)$, the left-hand side of this equation is a gradient of a scalar function. Thus, the right-hand side must also be a gradient of a scalar function. We denote it by $\nabla_\alpha \Psi$. Obviously, both sides of equation (5) are gradients in the particular case of a relativistic polytrope (which we now assume), i.e. when $e = np + \rho$ and $p = K\rho^{1+1/n}$. In this case the enthalpy H equals,

$$\begin{aligned} H &\equiv \int \frac{dp}{e + p} \\ &= \int \frac{dp}{(1+n)p + (p/K)^{n/n+1}} = \ln \left[(1+n) \frac{p}{\rho} + 1 \right] \approx (1+n) \frac{p}{\rho}. \end{aligned} \quad (6)$$

The last approximation is valid when the local sound speed $c_s \propto \sqrt{p/\rho}$ is negligible with respect to the speed of light $c = 1$. Substituting H into equation (5) and integrating, we find the Bernoulli equation in the form

$$\Psi + \frac{1}{2}\mathcal{E}_0^2 \mathcal{U} + (1+n) \frac{p}{\rho} = \text{const}. \quad (7)$$

We define the function $f = f(r, \theta)$ by

$$\frac{p}{\rho} = \frac{p_0}{\rho_0} f(r, \theta), \quad f(r, \theta) = 1 - \frac{1}{nc_{s0}^2} \left[\frac{\mathcal{E}_0^2}{2} (\mathcal{U} - \mathcal{U}_0) + \Psi \right], \quad (8)$$

where $c_{s0}^2 = (n+1)p/(n\rho)$ is the square of the sound speed at the torus centre. We now calculate the function f in the vicinity of the centre of the torus. For this purpose, we introduce new coordinates x and y by $dx = \sqrt{g_{rr}} dr/r_0$ and $dy = -\sqrt{g_{\theta\theta}} d\theta/r_0$, and the condition that $x = 0 = y$ at the torus centre. The metric coefficients in the above definition are evaluated at the centre of the torus. For the effective potential we obtain in terms of x, y ,

$$\mathcal{U} - \mathcal{U}_0 = \frac{r_0^2}{2} \left[\frac{1}{g_{rr}} \left(\frac{\partial^2 \mathcal{U}}{\partial r^2} \right)_0 x^2 + \frac{1}{g_{\theta\theta}} \left(\frac{\partial^2 \mathcal{U}}{\partial \theta^2} \right)_0 y^2 \right], \quad (9)$$

where all metric coefficients are evaluated at the torus centre. In the vicinity of the equilibrium point the potential Ψ can be expressed as

$$\Psi = -\frac{\mathcal{E}_0^2}{2g_{rr}} (g_{,r}^{tt} - \ell_0 g_{,r}^{t\phi}) \frac{1}{\ell_0} \left(\frac{\partial \ell}{\partial r} \right)_0 r_0^2 x^2. \quad (10)$$

Finally, by substitution into equation (8) and using equation (4), we obtain

$$f = 1 - \frac{1}{\beta^2} \left\{ \left[\bar{\omega}_r^2 - \frac{2\mathcal{R}_0}{\ell_0} \left(\frac{\partial \ell}{\partial r} \right)_0 \right] x^2 + \bar{\omega}_\theta^2 y^2 \right\}, \quad (11)$$

where

$$2\mathcal{R}_0 \equiv \frac{(g_{tt} + \Omega_0 g_{t\phi})^2}{\Omega_0^2 g_{rr}} (g_{,r}^{tt} - \ell_0 g_{,r}^{t\phi}), \quad \beta^2 \equiv \frac{(2nc_{s0}^2)}{(A_0^2 r_0^2 \Omega_0^2)}, \quad (12)$$

and $\bar{\omega}_r$ and $\bar{\omega}_\theta$ are ratios of the epicyclic frequencies to the orbital frequency Ω_0 at the centre of the torus.

The torus is *slender* when $\beta \ll 1$. The approximation used in (6) is obviously correct for slender tori. It is interesting to note that slender tori with constant angular momentum distributions $\ell(r, \theta) = \ell_0$ have elliptical cross-sections with semiaxes in the ratio of the epicyclic frequencies. The same is valid for Newtonian slender tori (Blaes *et al* 2005). In the particular case of the Newtonian $1/r$ gravitational potential, $\omega_r = \omega_\theta$ and slender tori have exactly circular cross-section (Madej and Paczyński 1977).

3. Perturbed equilibrium and epicyclic modes

3.1. Barotropic tori and the Papaloizou–Pringle equation

We consider small perturbations around the stationary and axially symmetric equilibrium of the torus (slender or not), in the form $\delta X(r, \theta, \phi, t) = \delta X_*(r, \theta) e^{i(m\phi - \omega t)}$. The equation that governs these perturbations was derived in Newtonian theory by Papaloizou and Pringle (1984) in terms of a single quantity W . Its general relativistic version has the form

$$W = \frac{\delta p}{u^t \rho (m\Omega - \omega)}. \quad (13)$$

For simplicity we assume for now that the fluid angular momentum $\ell = \text{const}$. We also assume that the flow is potential in the sense that

$$u^\alpha = \frac{\rho}{e + p} \nabla^\alpha \Phi, \quad (14)$$

where Φ is some scalar field. Linearizing this equation along with the normalization of the 4-velocity, we find that $\delta \Phi = iW$, so that

$$\delta u_\alpha = \frac{\rho}{e + p} [iW_{,\alpha} - u_\alpha u^t (m\Omega - \omega) W]. \quad (15)$$

Combining these with the perturbed continuity equation, $\nabla_\mu (\rho u^\mu) = 0$,

$$\nabla_\mu (\rho \delta u^\mu) = -i(m\Omega - \omega) u^t \delta \rho, \quad (16)$$

we arrive at the relativistic version of the Papaloizou–Pringle equation

$$\begin{aligned} \frac{1}{(-g)^{1/2}} \frac{\partial}{\partial x^i} \left[(-g)^{1/2} g^{ik} f^n \frac{\partial W}{\partial x^k} \right] - (m^2 g^{\phi\phi} - 2m\omega g^{t\phi} + \omega^2 g^{tt}) f^n W \\ + \frac{2nA^2 (m\Omega - \omega)^2}{\beta^2 A_0^2 r_0^2 \Omega_0^2} f^{n-1} W = 0, \end{aligned} \quad (17)$$

where g is the determinant of the metric and we used $c_s^2 \ll c^2$. In the slender torus limit $\beta \rightarrow 0$, this equation becomes

$$f \frac{\partial^2 W}{\partial \bar{x}^2} + n \frac{\partial f}{\partial \bar{x}} \frac{\partial W}{\partial \bar{x}} + f \frac{\partial^2 W}{\partial \bar{y}^2} + n \frac{\partial f}{\partial \bar{y}} \frac{\partial W}{\partial \bar{y}} + 2n(m - \nu)^2 W = 0, \quad (18)$$

where $\bar{x} \equiv x/\beta$, $\bar{y} \equiv y/\beta$ and $\nu \equiv \omega/\Omega_0$.

Equation (18) is identical in form with the nonrelativistic Papaloizou–Pringle equation for oscillations of the constant angular momentum slender torus (cf Blaes *et al* (2005)), so one could take advantage of Feynman’s reminder, ‘the same equations have the same solutions’. In the particular case $\omega_r = \omega_\theta$, the nonrelativistic Papaloizou–Pringle equation was *fully solved in exact analytic form* by Blaes (1985) who gave a complete set of its eigenmodes, with a complete analytic description of all eigenfunctions and all eigenfrequencies. Blaes’ solution cannot be directly applied to the relativistic Papaloizou–Pringle equation (17), because for relativistic tori $\omega_r \neq \omega_\theta$.

More recently, Blaes *et al* (2005) found that in the general Newtonian case (nonspherically symmetric potential, $\omega_r \neq \omega_\theta$), there are two particular solutions of the general Papaloizou–Pringle equation,

$$W_r \equiv C_r \bar{x} e^{i(m\phi - \omega_r t)} \quad W_z \equiv C_z \bar{y} e^{i(m\phi - \omega_\theta t)}, \quad (19)$$

with C_r, C_z being two arbitrary constants. We claim that they are also solutions of the relativistic Papaloizou–Pringle equation (18). One may check this by a direct substitution of solution (19) into equation (18).

The solution (19) is consistent with epicyclic modes. Indeed, the fluid velocity is spatially constant on the torus cross-sections, entirely radial in the case of the radial epicyclic mode W_r and entirely vertical in the case of the vertical epicyclic mode W_z . The radial and vertical oscillations are sinusoidal, with frequencies equal to the epicyclic frequencies ω_r and ω_θ .

3.2. Baroclinic tori

We assumed previously that the tori had constant specific angular momentum and were isentropic. This simplifies the perturbation equations enormously by removing the local restoring forces that give rise to internal inertial modes. We now consider more general configurations, and show that global epicyclic modes are still present even here.

The equilibrium of a relativistic baroclinic torus is described by equation (5), and the equations for axisymmetric perturbations are those of conservation of rest mass, momentum and adiabatic flow:

$$u^t \frac{\partial \delta \rho}{\partial t} + \rho g^{tt} \frac{\partial \delta u_t}{\partial t} + \rho g^{t\phi} \frac{\partial \delta u_\phi}{\partial t} = -\frac{1}{(-g)^{1/2}} \frac{\partial}{\partial r} \left[\frac{(-g)^{1/2} \rho \delta u_r}{g_{rr}} \right] - \frac{1}{(-g)^{1/2}} \frac{\partial}{\partial \theta} \left[\frac{(-g)^{1/2} \rho \delta u_\theta}{g_{\theta\theta}} \right], \quad (20)$$

$$u^t \frac{\partial \delta u_i}{\partial t} = \mathcal{E}[(g_{,i}^{tt} - \ell g_{,i}^{t\phi}) \delta u_t + (g_{,i}^{t\phi} - \ell g_{,i}^{\phi\phi}) \delta u_\phi] + \frac{P_{,i}}{(e+p)^2} (\delta e + \delta p) - \frac{1}{e+p} \delta p_{,i}, \quad (21)$$

$$u^t g^{\phi\phi} \frac{\partial \delta u_\phi}{\partial t} + u^t g^{t\phi} \frac{\partial \delta u_t}{\partial t} = \mathcal{E}^3 [g^{\phi\phi} g^{tt} - (g^{t\phi})^2] \delta \mathbf{u} \cdot \nabla l - \frac{u^t u^\phi + g^{t\phi}}{e+p} \frac{\partial \delta p}{\partial t}, \quad (22)$$

and

$$u^t \frac{\partial \delta p}{\partial t} - u^t c_s^2 \left(\frac{e+p}{\rho} \right) \frac{\partial \delta \rho}{\partial t} + \delta \mathbf{u} \cdot \nabla p - c_s^2 \left(\frac{e+p}{\rho} \right) \delta \mathbf{u} \cdot \nabla \rho = 0. \quad (23)$$

Here g is the determinant of the metric, $i = r$ or θ , and

$$c_s^2 = \left(\frac{\partial p}{\partial e} \right)_s = \frac{\rho}{e+p} \left(\frac{\partial p}{\partial \rho} \right)_s \quad (24)$$

is the adiabatic sound speed. These equations form a closed system when we include the perturbed normalization of the 4-velocity,

$$\delta u_t = -\Omega \delta u_\phi, \quad (25)$$

and a perturbed equation of state,

$$\delta e = \left(\frac{\partial e}{\partial p} \right)_\rho \delta p + \left(\frac{\partial e}{\partial \rho} \right)_p \delta \rho. \quad (26)$$

The relativistic perturbation equation for rest mass conservation can be simplified by neglecting derivatives of the metric in the slender torus limit. Equation (20) therefore becomes

$$u^t \frac{\partial \delta \rho}{\partial t} + \rho g^{tt} \frac{\partial \delta u_t}{\partial t} + \rho g^{t\phi} \frac{\partial \delta u_\phi}{\partial t} = -\frac{1}{g_{rr}} \frac{\partial}{\partial r} (\rho \delta u_r) - \frac{1}{g_{\theta\theta}} \frac{\partial}{\partial \theta} (\rho \delta u_\theta). \quad (27)$$

We now seek solutions of these equations in the slender torus limit in which δu_r and δu_θ are spatially constant. Equations (22), (23), (25) and (27) then imply that

$$\frac{\partial \delta p}{\partial t} = \frac{1}{D} \left\{ -\frac{1}{u^t} \delta \mathbf{u} \cdot \nabla p + \frac{(e+p)c_s^2 \ell \mathcal{E}^3}{(u^t)^2} [(g^{t\phi})^2 - g^{tt} g^{\phi\phi}] \delta \mathbf{u} \cdot \nabla \ell \right\}, \quad (28)$$

$$\begin{aligned} \frac{\partial \delta \rho}{\partial t} = & -\frac{1}{u^t} \delta \mathbf{u} \cdot \nabla \rho + \frac{1}{D} \left\{ -\frac{\rho \ell}{(e+p)(u^t)^3} (u^t u^\phi + g^{t\phi}) \delta \mathbf{u} \cdot \nabla p \right\} \\ & + \frac{1}{D} \left\{ \frac{\rho \ell \mathcal{E}^3}{(u^t)^2} [(g^{t\phi})^2 - g^{tt} g^{\phi\phi}] \delta \mathbf{u} \cdot \nabla \ell \right\}, \end{aligned} \quad (29)$$

and

$$\frac{\partial \delta u_\phi}{\partial t} = -\frac{1}{\Omega} \frac{\partial \delta u_t}{\partial t} = \frac{1}{D} \left\{ \frac{(u^t u^\phi + g^{t\phi})}{\mathcal{E} u^t (e+p) [(g^{t\phi})^2 - g^{tt} g^{\phi\phi}]} \delta \mathbf{u} \cdot \nabla p - \mathcal{E}^2 \delta \mathbf{u} \cdot \nabla \ell \right\}, \quad (30)$$

where

$$D \equiv 1 - \frac{c_s^2 \ell}{(u^t)^2} (u^t u^\phi + g^{t\phi}). \quad (31)$$

Now, in the slender torus limit, c_s^2 , p/ρ and p/e are all very small. Gradients in pressure and density also scale as one over the small thickness of the torus. Hence $D \rightarrow 1$ and equations (28)–(30) reduce to

$$\frac{\partial \delta p}{\partial t} = -\frac{1}{u^t} \delta \mathbf{u} \cdot \nabla p, \quad (32)$$

$$\frac{\partial \delta \rho}{\partial t} = -\frac{1}{u^t} \delta \mathbf{u} \cdot \nabla \rho, \quad (33)$$

and

$$\frac{\partial \delta u_\phi}{\partial t} = -\frac{1}{\Omega} \frac{\partial \delta u_t}{\partial t} = -\mathcal{E}^2 \delta \mathbf{u} \cdot \nabla \ell. \quad (34)$$

Differentiating equation (21) with respect to time, and using equations (26) and (32)–(34) to eliminate δe , δp , $\delta \rho$, δu_t and δu_ϕ , we obtain

$$\begin{aligned} (u^t)^2 \frac{\partial^2 \delta u_i}{\partial t^2} = & \delta \mathbf{u} \cdot \nabla \left[\frac{p_{,i}}{(e+p)} \right] - \frac{1}{u^t} \frac{\partial u^t}{\partial x^i} \delta \mathbf{u} \cdot \frac{\nabla p}{e+p} \\ & + u^t \mathcal{E}^3 (\delta \mathbf{u} \cdot \nabla \ell) [\Omega (g_{,i}^{tt} - \ell g_{,i}^{t\phi}) - g_{,i}^{t\phi} + \ell g_{,i}^{\phi\phi}]. \end{aligned} \quad (35)$$

The second term on the right-hand side of this equation is negligible compared to the others in the slender torus limit. Using the fact that the 4-acceleration vanishes at the pressure maximum, the following relationships are true in the slender torus limit:

$$u^t \mathcal{E}^3 \Omega \nabla \ell = \frac{1}{2} \nabla \mathcal{E}^2 = \frac{1}{2} \nabla (\mathcal{E}^2 - \mathcal{E}_0^2), \quad (36)$$

$$u^t \mathcal{E}^3 \ell \nabla \ell = \frac{1}{2} \nabla (\mathcal{E}^2 \ell^2) = \frac{1}{2} \nabla (\mathcal{E}^2 \ell^2 - \mathcal{E}_0^2 \ell_0^2), \tag{37}$$

and

$$u^t \mathcal{E}^3 (1 + \Omega \ell) \nabla \ell = \nabla (\mathcal{E}^2 \ell) = \nabla (\mathcal{E}^2 \ell - \mathcal{E}_0^2 \ell_0). \tag{38}$$

Hence equation (35) may be rewritten as

$$\begin{aligned} (u^t)_0^2 \frac{\partial^2 \delta u_i}{\partial t^2} &= \delta \mathbf{u} \cdot \nabla \left[\frac{P_{,i}}{(e+p)} \right] + \frac{1}{2} g_{,i}^{tt} \delta \mathbf{u} \cdot \nabla (\mathcal{E}^2 - \mathcal{E}_0^2) - g_{,i}^{t\phi} \delta \mathbf{u} \cdot \nabla (\mathcal{E}^2 \ell - \mathcal{E}_0^2 \ell_0) \\ &\quad + \frac{1}{2} g_{,i}^{\phi\phi} \delta \mathbf{u} \cdot \nabla (\mathcal{E}^2 \ell^2 - \mathcal{E}_0^2 \ell_0^2). \end{aligned} \tag{39}$$

In the slender torus limit, the metric derivatives may be absorbed inside the gradients, so that

$$\begin{aligned} (u^t)_0^2 \frac{\partial^2 \delta u_i}{\partial t^2} &= \delta \mathbf{u} \cdot \nabla \left[\frac{P_{,i}}{(e+p)} + \frac{1}{2} g_{,i}^{tt} (\mathcal{E}^2 - \mathcal{E}_0^2) - g_{,i}^{t\phi} (\mathcal{E}^2 \ell - \mathcal{E}_0^2 \ell_0) + \frac{1}{2} g_{,i}^{\phi\phi} (\mathcal{E}^2 \ell^2 - \mathcal{E}_0^2 \ell_0^2) \right] \\ &= -\frac{1}{2} \mathcal{E}_0^2 \delta \mathbf{u} \cdot \nabla (\nabla \mathcal{U})|_0, \end{aligned} \tag{40}$$

where the last equality comes from the equilibrium equation (5). Comparing equation (40) with equation (4), we see that we once again have radial and vertical modes which oscillate at the respective relativistic epicyclic frequencies.

4. Discussion and conclusions

We have shown that the equations of motion of a fluid slender torus in axially symmetric, stationary spacetimes admit small oscillations occurring at the epicyclic frequencies¹⁰. For an initially axisymmetric torus, these two oscillations are exact solutions of the fluid perturbation equations in the limit of vanishing sound speed (when compared to the orbital frequency), and therefore represent two modes of oscillation of a slender torus. One of these modes corresponds to a uniform vertical displacement of the torus at the vertical epicyclic frequency, the other to a purely radial motion at the radial epicyclic frequency.

In this paper (paper II) we considered *linear* deviations from circular geodesic motion. In the linear approximation the two epicyclic modes are uncoupled and independent. In reality, nonlinear terms would inevitably couple the two modes. In the other papers of this series we discuss how the coupling leads naturally to the 3/2 resonance. Because the 3/2 resonance is the main motivation for the whole series, we summarize here an idea as to how the resonance is excited and why the ratio 3/2 is a natural consequence of strong gravity.

For Newtonian slender tori, paper III shows that (slightly) away from the exact slender torus limit $\beta = 0$ (which corresponds to $c_s = 0$), the frequencies of vertical and radial oscillations of fluid are

$$(\omega_\theta^2)_f = \omega_\theta^2 - A_z(n) c_s^2, \quad (\omega_r^2)_f = \omega_r^2 - A_r(n) c_s^2, \tag{41}$$

where $A_z(n)$ and $A_r(n)$ are explicit positive functions of the polytropic constant n . In paper I (Kluźniak and Abramowicz 2002) we argued that the presence of c_s^2 in both frequencies

¹⁰ This is a specific example of a more general property. Consider an isolated fluid body, moving in a fixed spacetime. Synge (1960) proved that inside the world tube of the body a geodesic line exists. For the fluid tori considered here, this is the geodesic circle at the pressure maximum. A small perturbation of the body results in a perturbed world tube that also contains a geodesic line, close to the original one. The spacetime distance between them is given by the geodesic deviation equation that in the special case of a geodesic circle (2) in the spacetime (1) reduces to two uncoupled epicyclic oscillations with frequencies (4). Global oscillations of fluids can have frequencies slightly different from the geodesic epicyclic frequencies, as equation (41) shows for nonslender tori. See also Biesiada (2003), Colistete *et al* (2002) and Kerner *et al* (2001).

provides a pressure coupling between these two modes and that, accordingly, one may consider the vertical mode as being described by the Mathieu type equation,

$$\frac{d^2}{dt^2}\delta\theta + \omega_\theta^2[1 + A \cos(\omega_r t)]\delta\theta = 0. \quad (42)$$

It is well known that a parametric resonance occurs when

$$\omega_r = \frac{2\omega_\theta}{n}, \quad n = 1, 2, 3, \dots \quad (43)$$

For black holes, $\omega_\theta > \omega_r$, and therefore the smallest value of n consistent with the resonance is $n = 3$ (paper I), which corresponds to the 3/2 ratio of the vertical and radial epicyclic frequencies.

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Note added in proof. J Homan of MIT reported at the AAS meeting on 9 January 2006 that GRO J1655-40 showed in 2005 twin peak QPOs with frequencies of 300 Hz and 450 Hz, i.e. the same as observed by T Strohmayer of NASA nine years ago. Both the remarkable stability of the QPO frequencies after nine years, and their 3/2 ratio, are in accord with the orbital resonance model discussed in this paper.

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