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The autoparametric 3:2 resonance in conservative systems

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Abstract. In the resonance model, high-frequency quasi-periodic oscillations (QPOs) are supposed to be a consequence of nonlinear resonance between modes of oscillations occurring within the innermost parts of an accretion disk. Several models with a prescribed mode–mode interaction were proposed in order to explain the characteristic properties of the resonance in QPO sources. In this paper, we examine nonlinear oscillations of a system having a quadratic nonlinearity and we show that this case is particularly relevant for QPOs. We present a very convenient way how to study autoparametric resonances of a fully general system using the method of multiple scales. We concentrate to conservative systems and discuss their behavior near the $3:2$ parametric resonance.

Key words: nonlinear resonance – perturbation methods – multiple scales

1. Introduction

In the resonance model (Abramowicz & Kluźniak 2001; Kluźniak & Abramowicz 2000; Kato 2003), there is a natural and attractive possibility of explaining the observed rational ratios of high-frequency QPOs as a consequence of nonlinear coupling between different modes of accretion disk oscillations. The idea has been pursued in several papers (recently, e.g. Abramowicz et al. 2003; Rebusco 2004).

Specific models invoke particular physical mechanisms. Some models can be almost immediately comprehended as distinct realizations of the general approach discussed here – for example, various formulations of the orbiting spot model (Schnittman & Bertchinger 2004) or the models, where QPOs are produced by the magnetically driven resonance in a diamagnetic accretion disk (Lai 1999) – while other seem to be more distant from the view presented herein – e.g. the transition layer model (Titarchuk 2002), an interesting idea of p-mode oscillations of a small accretion torus (Rezzola et al. 2003) or the model of blobs in an accretion disc (see e.g. Karas 1999; Li & Narayan 2004, and references cited therein). Also in this context, Kato (2004) discussed the resonant interaction between waves propagating in a warped disk, including their rigorous mathematical description. Instead of pursuing a specific model, here we keep the discussion as general as possible, aiming to implement the formalism of multiple scales. Indeed, we show that there is unquestionable appeal in this approach which offers some additional insight into generic properties of resonant oscillations.

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Some properties of an accretion disk oscillations can be discussed within the epicyclic approximation of a test particle on a circular orbit near equatorial plane. Suppose that angular momentum of the particle is fixed to a value ℓ. The effective potential $U_\ell(r,\theta)$ has a minimum at radius r_0, corresponding to the location of the stable circular orbit. An observer moving along this orbit measures radial, vertical and azimuthal epicyclic oscillations of a particle nearby. Since the angular momentum of the particle is conserved, only two of them – radial and vertical – are independent. The epicyclic frequencies can be derived from the geodesic equations expanded to the linear order in deviations $\delta r = r - r_0$ and $\delta\theta = \theta - \pi/2$ from the circular orbit. We get two independent second-order differential equations describing two uncoupled oscillators with frequencies ω_r and ω_θ, which are given by the second derivatives of effective potential $U_\ell(r,\theta)$. In Newtonian theory, ω_r and ω_θ are equal to the Keplerian orbital frequency $\Omega_{\rm K}$. This is in tune with the fact that orbits of particles are planar and closed curves. The degeneracy between two epicyclic frequencies can be seen as a result of scale-freedom of the Newtonian gravitational potential (Abramowicz & Kluźniak 2003). In Schwarzschild geometry this freedom is broken by introducing the gravitational radius $r_g = 2GM/c^2$. The degeneracy between the vertical epicyclic and the orbital frequencies is related to spherical symmetry of the gravitational potential, which assures the existence of planar trajectories of particles. All three frequencies are different in the vicinity of a rotating Kerr black hole.

In addition, when nonlinear terms of geodesic equations are included, the two oscillations in r and θ directions become coupled and variety of new phenomena connected to nonlinear nature of the equations appear. This rich phenomenology includes frequency shift of observed frequencies with respect to eigenfrequencies, presence of higher harmonics and subharmonics, drifts and parametric resonance. The first three are connected to nonlinear oscillations of each mode and the last one comes from the coupling between two modes.

2. Expansion via multiple scales

Let us study nonlinear oscillations of the system having two degrees of freedom, i.e., the coordinate perturbations δr and $\delta\theta$. The oscillations are described by two coupled differential equations of the very general form

$$\ddot{\delta r} + \omega_r^2\, \delta r \;=\; \omega_r^2\, f_r(\delta r, \delta\theta, \dot{\delta r}, \dot{\delta\theta}), \tag{1}$$

$$\ddot{\delta\theta} + \omega_\theta^2\, \delta\theta \;=\; \omega_\theta^2\, f_\theta(\delta r, \delta\theta, \dot{\delta r}, \dot{\delta\theta}). \tag{2}$$

Suppose that the functions f_r and f_θ are nonlinear, i.e., their Taylor expansions start in the second order. Another assumption is that these functions are invariant under reflection of time (i.e., the Taylor expansion does not contain odd powers of time derivatives of δr and $\delta\theta$). As we see later, this assumption is related to the conservation of energy in the system. Many authors studied such systems with a particular form of functions f and g (Nayfeh & Mook 1979), however, in this paper we keep discussion fully general.

We seek the solutions of the governing equations in the form of the multiple-scales expansions (Nayfeh & Mook 1979)

$$\delta r(t,\epsilon) = \sum_{n=1}^{4} \epsilon^n r_n(T_\mu), \quad \delta\theta(t,\epsilon) = \sum_{n=1}^{4} \epsilon^n \theta_n(T_\mu), \tag{3}$$

where several time scales T_μ are introduced instead of the physical time t,

$$T_\mu \equiv \epsilon^\mu t, \;\; \mu = 0, 1, 2, 3. \tag{4}$$

The time scales are treated as independent. It follows that instead of the single time derivative we have an expansion of partial derivatives with respect to the T_μ

$$\frac{\mathrm{d}}{\mathrm{d}t} \;=\; D_0 + \epsilon D_1 + \epsilon^2 D_2 + \epsilon^3 D_3 + \mathcal{O}(\epsilon^4), \tag{5}$$

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \;=\; D_0^2 + 2\epsilon D_0 D_1 + \epsilon^2 (D_1^2 + 2 D_0 D_2) + 2\epsilon^3 (D_0 D_3 + D_1 D_2) + \mathcal{O}(\epsilon^4), \tag{6}$$

where $D_\mu = \partial/\partial T_\mu$.

We expand the nonlinear functions f_r and f_θ are can be expanded into the Taylor series and then we substitute the expansions (3), (5) and (6). Finally, we compare the coefficients of the same powers of ϵ on both sides in the resulting couple of equations. This way we get a set of *linear* second-order differential equations that can be solved successively – the lower-order terms of the expansion (3) appear as forcing terms on the right-hand sides in the equations for the higher order approximations.

In the first order we obtain equations corresponding to the linear approximation

$$(D_0^2 + \omega_r^2) r_1 = 0, \quad (D_0^2 + \omega_\theta^2)\theta_1 = 0 \tag{7}$$

with the solutions

$$x_1 = A_r(T_1, T_2, T_3) e^{i\omega_r T_0} + \mathrm{cc}, \tag{8}$$

$$\theta_1 = A_\theta(T_1, T_2, T_3) e^{i\omega_\theta T_0} + \mathrm{cc}. \tag{9}$$

The complex amplitudes $\widehat{A}_r$ and A_θ generally depend on the higher time-scales.

The solutions (8) and (8) substituted into the quadratic terms in the right-hand side of the second-order differential equations produces terms that oscillates with frequencies $2\omega_r$, $2\omega_\theta$ and $\omega_\theta \pm \omega_r$. When the frequency ratio ω_r/ω_θ is far from $1:2$ and $2:1$ the solutions r_2 and θ_2 describe higher harmonics to the linear-order oscillations r_1 and θ_1. Hence, the presence of higher harmonics in the power-spectra is a general signature of nonlinear oscillations. Their frequencies and relative strengths with respect to the main oscillations could provide us useful informations about nonlinearities in the system.

In addition, the right hand sides of the second order equations contain terms proportional to $e^{i\omega_r T_0}$ and $e^{i\omega_\theta T_0}$ that oscillates with the same frequency as the eigenfrequency of the oscillators. These terms produces secular grow of the amplitudes of the second-order approximations r_2 and θ_2 and causes nonuniform expansions (3). Eliminating them we get the *solvability conditions* for the complex amplitudes $A_r(T_1, T_2, T_3)$ and $A_\theta(T_1, T_2, T_3)$ that give us the evolution of the system on longer time-scales (Nayfeh & Mook 1979).

When the eigenfrequencies are in 1:2 or 2:1 ratio we observe qualitatively different behavior related to the *autoparametric resonance*. In that case the right hand sides contains additional secular terms and the solvability conditions take different form. Different resonances occur in different orders of approximation. The possible resonances in the third order are 1:3, 1:1 and 3:1 and 1:4, 3:2, 2:3 and 4:1 in the fourth order[1] However, if the governing equations remain unchanged under the transformation $\delta\theta \to -\delta\theta$ (i.e., the system is reflection symmetric) the only autoparametric resonances that exists in the system are 1:2, 1:1, 1:4 and 3:2 (Rebusco 2004)

3. The 3:2 autoparametric resonance

Let us study oscillations of the conservative system eigenfrequencies of which are close to 3:2. The time behavior of the observed frequencies $\omega_r^\star$ and $\omega_\theta^\star$ and amplitudes a_r and a_θ of the oscillations can be found from the solvability conditions imposed on the complex amplitudes $A_r(T_1, T_2, T_3)$ and $A_\theta(T_1, T_2, T_3)$ (Horák et al. 2005)

$$D_1 A_r \;=\; D_2 A_\theta = 0, \tag{10}$$

$$D_2 A_r \;=\; -\frac{i\omega_r}{2}\left[\kappa_r |A_r|^2 + \kappa_\theta |A_\theta|^2\right] A_r, \tag{11}$$

[1] The ratio $n:m$ refers to the eigenfrequency ratio $\omega_\theta : \omega_r$.

$$D_2 A_\theta = -\frac{i\omega_\theta}{2}\left[\lambda_r |A_r|^2 + \lambda_\theta |A_\theta|^2\right] A_\theta, \quad (12)$$

$$D_3 A_r = -\frac{i}{2}\omega_r \alpha (A_r^2)^* A_\theta^2 e^{-i(\sigma_2 T_2 + \sigma_3 T_3)}, \quad (13)$$

$$D_3 A_\theta = -\frac{i}{2}\omega_\theta \beta A_r^3 A_\theta^* e^{i(\sigma_2 T_2 + \sigma_3 T_3)}. \quad (14)$$

In the fourth order we eliminate also terms which become secular in when $3\omega_r \approx 2\omega_\theta$. For this purpose we describe the vicinity of the resonance by the detuning parameters σ_2 and σ_3 introduced according to

$$3\omega_r = 2\omega_\theta + \epsilon^2 \sigma_2 + \epsilon^3 \sigma_3. \quad (15)$$

The term $\epsilon\sigma_1$ is missing, because the complex amplitudes depends only on the second and the third time-scales. The solvability conditions describe the evolution of the system in the most general way: the real parameters α, β, κ_r, κ_θ, λ_r and $\lambda\theta$ are given by the coefficients of the Taylor expansion of the nonlinear functions f_r and f_θ.

Since A_r and A_θ are complex, the conditions (10)–(14) represent 12 real equations. However few of them are trivial. By substituting the polar forms $\epsilon A_r = \frac{1}{2} a_r e^{i\phi_r}$ and $\epsilon A_\theta = \frac{1}{2} a_\theta e^{i\phi_\theta}$, separating real and imaginary parts and introducing the unique time t the number of the equations can be reduced to the four,

$$\dot{a}_\rho = \frac{\alpha\omega_r}{16}\, a_\rho^2\, a_\theta^2\, \sin\gamma, \quad (16)$$

$$\dot{a}_\theta = -\frac{\beta\omega_\theta}{16}\, a_\rho^3\, a_\theta\, \sin\gamma, \quad (17)$$

$$\dot{\phi}_\rho = -\frac{\omega_r}{2}\left[\kappa_r\, a_\rho^2 + \kappa_\theta\, a_\theta^2\right] - \frac{\alpha\omega_r}{16}\, a_\rho\, a_\theta^2 \cos\gamma, \quad (18)$$

$$\dot{\phi}_\theta = -\frac{\omega_\theta}{2}\left[\lambda_r\, a_\rho^2 + \lambda_\theta\, a_\theta^2\right] - \frac{\beta\omega_\theta}{16} a_\rho^3 \cos\gamma, \quad (19)$$

where we introduced the phase function $\gamma(t) = 2\phi_\theta(t) - 3\phi_r(t) - \sigma t$ and the unique detuning parameter $\sigma = \epsilon^2\sigma_2 + \epsilon^3\sigma_3$. The Eqs. (16) and (17) describe the slow evolution of the amplitudes of the oscillations, additional long-term behavior of the oscillation phases is given by Eqs. (18) and (19). These equations give us the frequency-shift of the observed frequencies $\omega_r^\star$ and $\omega_\theta^\star$ with respect to the eigenfrequencies ω_r and ω_θ, respectively,

$$\omega_r^* = \omega_r + \dot{\phi}_r, \quad \omega_\theta^* = \omega_\theta + \dot{\phi}_\theta. \quad (20)$$

The two Eqs. (18) and (19) can be replaced by a single differential equation for the phase function,

$$\dot{\gamma} = -\sigma + \frac{\omega_\theta}{4}\left[\mu_r a_r^2 + \mu_\theta a_\theta^2 + \frac{a_r}{2}\left(\alpha a_\theta^2 - \beta a_r^2\right)\cos\gamma\right], \quad (21)$$

where we used the fact that near the resonance $\omega_r \approx (2/3)\omega_\theta$ and we defined $\mu_r = \kappa_r - \lambda_r$ and $\mu_\theta = \kappa_\theta - \lambda_\theta$.

3.1. Steady-state solutions

Steady-state solutions are characterized by constant amplitudes and frequencies of oscillations. Such solutions represent singular points of the system governed by Eqs. (16), (17) and (21).

It is obvious from Eqs. (16) and (17) that the condition $\dot{a}_r = \dot{a}_\theta = 0$ can be satisfied (with nonzero amplitudes) only if $\sin\gamma = 0$ (identically at all times), and thus also $\dot{\gamma} = 0$. In that case Eq. (21) transforms to the algebraic equation

$$\frac{\sigma}{\omega_\theta} = \frac{1}{4}\left[\mu_r a_r^2 + \mu_\theta a_\theta^2 \pm \frac{a_r}{2}\left(\alpha a_\theta^2 - \beta a_r^2\right)\right]. \quad (22)$$

The left-hand side can be expressed using the eigenfrequency ratio $R = \omega_\theta/\omega_r$ as

$$\frac{\sigma}{\omega_\theta} = -\frac{2}{R}\left(R - \frac{3}{2}\right). \quad (23)$$

Then we get

$$R = \frac{3}{2} - \frac{3}{16}\left(\mu_r a_r^2 + \mu_\theta a_\theta^2\right) \pm \frac{3}{32} a_r \left(\alpha a_\theta^2 - \beta a_r^2\right), \quad (24)$$

where we neglected terms of order a^4. Note that the lowest correction to eigenfrequencies is of order of a^2 – for given amplitudes a_r, a_θ steady-state oscillations occur when the ratio of eigenfrequencies departs from $3/2$ by deviation of order of a^2.

The relation between observed frequencies of oscillations $\omega_r^\star$, $\omega_\theta^\star$ and eigenfrequencies ω_r, ω_θ are given by the time derivative of phases ϕ_r and ϕ_θ. We can find simple relation between observed frequencies and the phase function

$$\begin{aligned} 3\omega_r^\star - 2\omega_\theta^\star &= 3\omega_r - 2\omega_\theta + (3\dot{\phi}_r - 2\dot{\phi}_\theta) \\ &= \sigma + (3\dot{\phi}_r - 2\dot{\phi}_\theta) = -\dot{\gamma}. \end{aligned} \quad (25)$$

For steady state solutions $\dot{\gamma} = 0$, and thus observed frequencies are adjusted to *exact* $3:2$ ratio even if eigenfrequencies depart from it.

3.2. Integrals of motion

Behavior of the system is described by three variables $a_r(t)$, $a_\theta(t)$ and $\gamma(t)$ and three first-order differential equations (16), (17) and (21). However, the number of differential equations can be reduced to one because it is possible two find two integrals of motion of the system. Our discussion will be analogical to case of $1:2$ resonance of systems with quadratic nonlinearity, as examined by Nayfeh & Mook (1979).

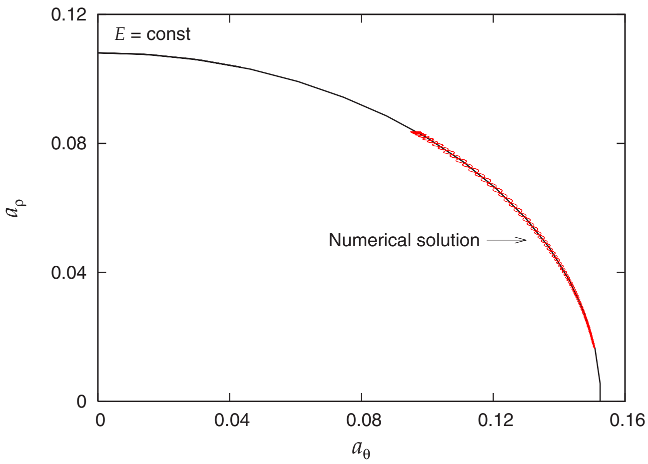

Fig. 1. Comparison between an analytical constraint (27) and the corresponding numerical solution of the system Abramowicz et al. (2003). Each point corresponds to the amplitudes of the oscillations at a particular time. On the other hand, from the discussion of Eq. (27) we know that these points must lay on an ellipse, whose shape is determined by the multiple-scales method.

Consider Eqs. (16) and (17). Eliminating $\sin\gamma$ from both equations we find

$$\frac{\mathrm{d}}{\mathrm{d}t}(a_r^2 + \nu a_\theta^2) = 0 \tag{26}$$

and thus

$$a_r^2 + \nu a_\theta^2 = \mathrm{const} \equiv E, \tag{27}$$

where we defined

$$\nu = \frac{\alpha\omega_r}{\beta\omega_\theta} \approx \frac{2\alpha}{3\beta}. \tag{28}$$

When $\nu > 0$, the both amplitudes of oscillations are bounded. The curve $[a_r(t), a_\theta(t)]$ is a segment of an ellipse. The constant E is proportional to the energy of the system. On the other hand, when $\nu < 0$, one amplitude of oscillations can grow without bounds while the second amplitude vanishes. This case corresponds to the presence of an regenerative element in the system (Nayfeh & Mook 1979). The corresponding curve in the (a_r, a_θ) plane is a hyperbola. In further discussion we assume that $\nu > 0$.

In order to verify that the the energy of the system is conserved, we numerically integrated governing Eq. (1) and (2) for the one particular system discussed by Abramowicz et al. (2003). The comparison is in Fig. 1. The numerical and analytical results are in a very good agreement.

The second integral of motion is found in following way. Let us multiply the Eq. (21) by a_θ. Then we obtain

$$a_\theta\dot\gamma = -\sigma a_\theta + \frac{\omega_\theta}{4}\mu_r a_r^2 a_\theta + \frac{\omega_\theta}{4}\mu_\theta a_\theta^3 + \frac{\omega_\theta}{8}\alpha a_r a_\theta^3\cos\gamma - \frac{\omega_\theta}{8}\beta a_r^3 a_\theta\cos\gamma. \tag{29}$$

Changing the independent variable from t to a_θ and multiplying the whole equation by $\mathrm{d}a_\theta$ we find

$$a_r^3 a_\theta^2\mathrm{d}(\cos\gamma) + \frac{8\sigma}{\beta\omega_\theta}\mathrm{d}(a_\theta^2) - \frac{4\mu_r}{\beta}a_r^2 a_\theta\mathrm{d}(a_\theta^2) - \frac{\mu_\theta}{\beta}\mathrm{d}(a_\theta^4) - -\frac{2\alpha}{\beta}a_r a_\theta^3\cos\gamma\mathrm{d}a_\theta + 2a_r^3 a_\theta\cos\gamma\mathrm{d}a_\theta = 0. \tag{30}$$

The Eq. (27) implies

$$a_\theta\mathrm{d}a_\theta = -\frac{a_r\mathrm{d}a_r}{\nu}. \tag{31}$$

With the aid of this relation the Eq. (30) takes the form

$$3a_r^2 a_\theta^2\cos\gamma\mathrm{d}a_r + 2a_r^3 a_\theta\cos\gamma\mathrm{d}a_\theta + a_r^3 a_\theta^2\mathrm{d}(\cos\gamma) + +\frac{8\sigma}{\beta\omega_\theta}\mathrm{d}(a_\theta^2) + \frac{\mu_r}{\beta\nu}\mathrm{d}(a_r^4) - \frac{\mu_\theta}{\beta}\mathrm{d}(a_\theta^4) = 0. \tag{32}$$

The first three terms express the total differential of function $-a_r^3 a_\theta^2\cos\gamma$. Hence, the above equation can be arranged to the form

$$\mathrm{d}\left(a_r^3 a_\theta^2\cos\gamma + \frac{8\sigma}{\beta\omega_\theta}a_\theta^2 + \frac{\mu_r}{\beta\nu}a_r^4 - \frac{\mu_\theta}{\beta}a_\theta^4\right) = 0. \tag{33}$$

In other words,

$$a_r^3 a_\theta^2\cos\gamma + \frac{8\sigma}{\beta\omega_\theta}a_\theta^2 + \frac{\mu_r}{\beta\nu}a_r^4 - \frac{\mu_\theta}{\beta}a_\theta^4 = \mathrm{const} \equiv L \tag{34}$$

is another integral of Eqs. (16), (17) and (21).

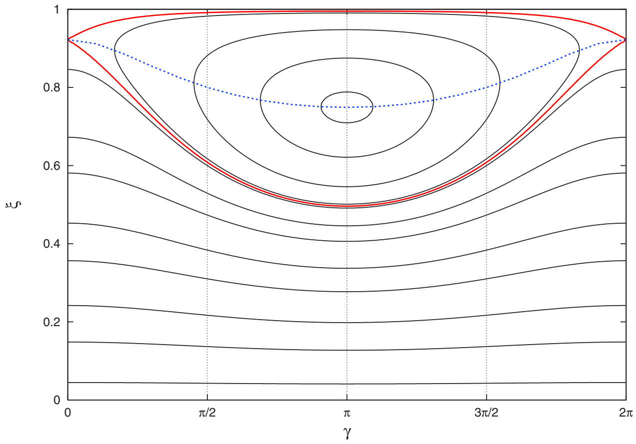

Fig. 2. Example of the (γ, ξ)-plane for the system close to the 3:2 resonance. The oscillations are coupled by nonlinear functions f_ρ and f_θ [see Eqs. (1) and (2)]. These functions give us values of the constants α, β, κ_r, κ_θ. λ_r and λ_θ. The thicksolid line is separatrix dividing the librating and circulating trajectories The blue dotted line connects points where $\dot\gamma = 0$.The example is for values $\alpha = \beta = \kappa_r = \lambda_\theta = 1$, $\kappa_\theta = \lambda_\theta = 2$, $\mathcal{E} = 0.1$ and $\sigma = -0.165$.

3.3. Analytical results

Knowing two integrals of motion, we should be able to find one differential equation which governs the time-evolution of the system.

First, the amplitudes a_r and a_θ are not independent because they are related by Eq. (27). To satisfy this relation, let us define new variable $\xi(t)$ by

$$a_r^2 = \xi E, \quad a_\theta^2 = (1-\xi)\frac{E}{\nu}. \tag{35}$$

Later on, we derive the evolution equation for $\xi(t)$. For present moment, we ignore the time dependence by considering projections of solutions into the (γ, ξ)-plane. For a fixed energy E of oscillations, the system follows curves of constant L. Hence, the projections of solutions into the (γ, ξ)-plane are given by equation

$$L(\gamma, \xi) = \mathrm{const}. \tag{36}$$

An example of the phase-plane is given in Fig. 2. There are two types of trajectories $[\xi(t), \gamma(t)]$: the *circulating* trajectories take the full range $0 \le \gamma(t) \le 2\pi$ and the *librating* trajectories that are confined in the smaller range $\gamma_1 \le \gamma(t) \le \gamma_2$. The turning points on the librating trajectories correspond to $\gamma = \gamma_1$ and $\gamma = \gamma_2$. This division has an interesting consequences with respect to the frequencies of the resonant oscillations. According to the relation (25) the observed frequencies are in exact 3:2 ratio when the system pass through these points. On the other hand, the circulating trajectories do not contain any turning points and the ratio of observed frequencies are always above or bellow 3:2.

The equation describing the evolution of $\xi(t)$ can be derived in the following way. Let us multiply Eq. (16) by $2a_r$ and integrate it. We obtain

$$\frac{\mathrm{d}(a_r^2)}{\mathrm{d}t} = \frac{\alpha}{8}\omega_r a_r^3 a_\theta^2\sin\gamma. \tag{37}$$

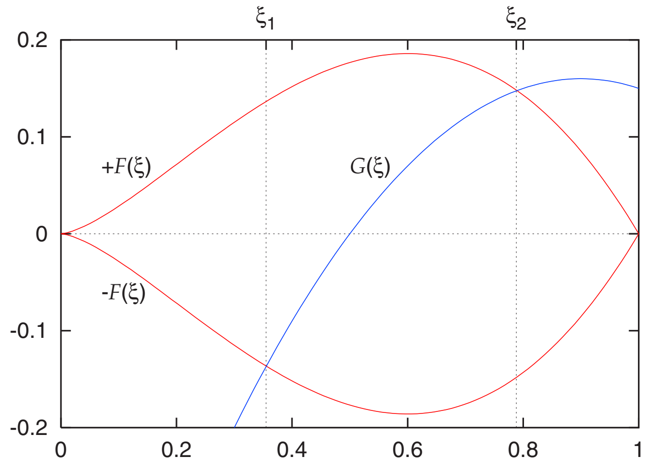

Fig. 3. The functions $\pm F(\xi) = \pm(1-\xi)\xi^{3/2}$ and the quadratic function $G(\xi)$ which second powers are first and second terms on the right-hand side of the Eq. (40). The behavior of the system corresponds to ξ in the interval $[\xi_1, \xi_2]$ (denoted by the two dotted vertical lines) where the condition $|F(\xi)| \geq |G(\xi)|$ is satisfied.

Then we express a_r^2 using ξ, and square it. We find

$$\left(\frac{8E}{\alpha\omega_r}\right)^2 \dot{\xi}^2 = \left(a_r^3 a_\theta^2 \sin\gamma\right)^2. \tag{38}$$

The right-hand side of this equation can be expressed using Eq. (34)

$$\left(a_r^3 a_\theta^2 \sin\gamma\right)^2 = \left(a_r^3 a_\theta^2\right)^2 - \left(L - \frac{8\sigma}{\beta\omega_\theta} a_\theta^2 - \frac{\mu_r}{\beta\nu} a_r^4 + \frac{\mu_\theta}{\beta} a_\theta^4\right)^2. \tag{39}$$

After the substitution into the Eq. (38) and using the relations (35), we get

$$\frac{1}{E^3}\left(\frac{8}{\beta\omega_\theta}\right)^2 \dot{\xi}^2 = (1-\xi)^2\xi^3 - \frac{\nu^2}{E^5}\Big[L - \frac{8\sigma E}{\beta\nu\omega_\theta}(1-\xi) - \frac{\mu_r E^2}{\beta\nu}\xi^2 + \frac{\mu_\theta E^2}{\beta\nu^2}(1-\xi)^2\Big]^2. \tag{40}$$

The equation of motion has very familiar form

$$\mathcal{K}^2\dot{\xi}^2 = F^2(\xi) - G^2(\xi), \tag{41}$$

where the $\mathcal{K}^2$ is a positive constant, $F(\xi) = (1-\xi)\xi^{3/2}$ and $G(\xi)$ is a quadratic function coefficients of which depend on initial conditions through E and L. For example, the equation with an effective potential, which governs motion of a test particle around a massive body, has the same form. Hence, the following discussion is identical as in that case.

In general, the motion occurs only when $\dot{\xi}^2$ is positive and thus for ξ that satisfy $|F(\xi)| \geq |G(\xi)|$. The turning points, where $\dot{\xi}$ changes its signature, are determined by the condition

$$|F(\xi)| = |G(\xi)|. \tag{42}$$

The functions $\pm F(\xi)$ and $G(\xi)$ are plotted in Fig. 3. Generally, the function G intersects the functions $\pm F$ in two points that corresponds to $\xi(t)$ oscillating between the two bounds ξ_1 and ξ_2 given by condition (42). The radial and vertical mode of oscillations periodically exchanges the energy. The amount of exchanged energy is given by $\Delta E/E = \xi_2 - \xi_1$. For some rather particular values of L and E only one intersection of $\pm F$ and G exists (the function $G(\xi)$ touch one of the functions $\pm F(\xi)$. Then the oscillations of the system correspond to the steady-state solutions discussed above.

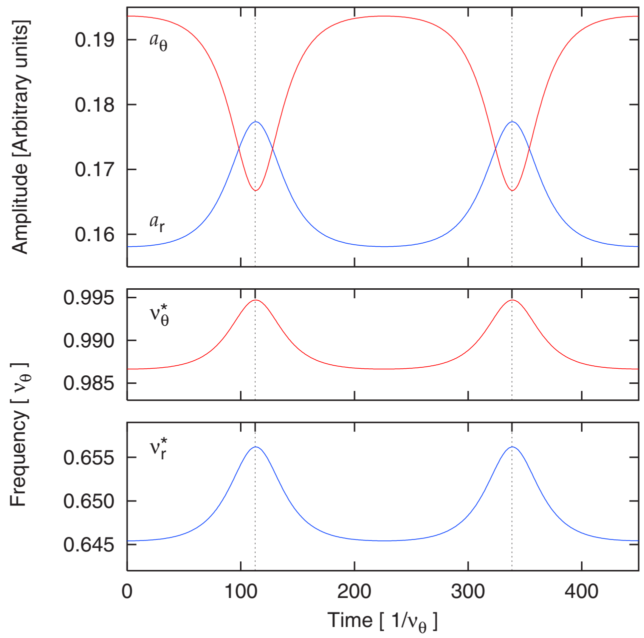

Fig. 4. Time evolution of the amplitudes (top panel) and the observed frequencies, $\nu_\theta^\star = \omega_\theta^\star/(2\pi)$ (middle panel) and $\nu_r^\star = \omega_r^\star/(2\pi)$ (bottom panel). All quantities are rescaled with respect the higher eigenfrequency ν_θ. Amplitudes of the oscillations are anti-correlated because the energy is conserved. Observed frequencies are correlated because the system is in parametric resonance.

The period of the energy exchange can be find by integration of the Eq. (40)

$$T = \frac{16}{\beta\omega_\theta} E^{-3/2} \int_{\xi_1}^{\xi_2} \frac{d\xi}{\sqrt{F^2(\xi) - G^2(\xi)}}. \tag{43}$$

This integral can be roughly approximated as

$$T \sim \frac{16\pi}{\beta\omega_\theta} E^{-3/2}. \tag{44}$$

However, near the steady state the period becomes much longer.

The observed frequencies $\omega_r^\star$ and $\omega_\theta^\star$ are given by relations (25). They depend on squares of amplitudes a_r and a_θ. Since both a_r^2 and a_θ^2 depend linearly on $\xi(t)$, also observed frequencies are linear functions of ξ and are linearly correlated. The slope of this correlation $\omega_\theta^\star = K\omega_r^\star + Q$ is independent of the energy of oscillations and is given only by parameters of the system,

$$K = \frac{\omega_\theta}{\omega_r}\frac{\lambda_r\nu - \lambda_\theta}{\kappa_r\nu - \kappa_\theta}. \tag{45}$$

The slope of the correlation differs from $3:2$, however the observed frequencies are still close to it.

3.4. Numerical results

The Eqs. (16), (17) and (21) were solved numerically using the fifth-order Runge-Kutta method with an adaptive step size. One of the solutions is shown in Fig. 4. The top panel of the figure shows the time behavior of the amplitudes of the two modes of oscillations. Since energy of the system is constant, amplitudes are anticorrelated and the two modes are continuously exchanging energy between each other. The middle and the bottom panels show the two observed frequencies that are mutually correlated. They are also correlated to one of the amplitudes. The frequency ratio varies with time and it differs from the exact 3:2 ratio, however, it always remains very close to it. The numerical solution is in agreement with the general results obtained analytically in the previous section.

4. Conclusions

Although this paper was originally motivated by observations and models connected to high-frequency QPOs, our results are very general and can be applied to any system with governing equations of the form (1) and (2).

The main result of these calculations is our prediction of low-frequency modulation of the amplitudes and frequencies of oscillations. The characteristic timescale of the modulation is approximately given by Eq. (44).

Because of the generality of our approach this fact have an interesting consequences in the context of QPO nature of which are unknown. Our result can be summarized in the following way: If the two quasiperiodic oscillations observed close to 3:2 ratio are produced by the autoparametric resonance the frequencies and amplitudes of oscillations should be periodically modulated. This modulation appears as a separate peak at the modulation frequency and as side-bands to the main (linear) oscillation. In a separate paper by Horák et al. (2004) we pointed to possible connection of this modulation with the 'normal branch oscillations' (NBOs) that are often present together with QPOs. Specifically, we suggest that the correlation between the higher frequency and the lower amplitude, evident in Fig. 4, is the same as was recently reported in Sco X-1 by Yu et al. (2001).

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