On polarization of light scattered on hot electron clouds

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Abstract

In this note we study polarization properties of radiation scattered on relativistic electrons in a hot cloud. The electron distribution is considered isotropic in the cloud comoving frame. We derive simple formulae for frequency-integrated Stokes parameters I, Q and U of the scattered radiation. The last parameter V vanishes because the resulting polarization is linear. The Stokes parameters are evaluated in the polarization frame comoving with the cloud, whose one basis vector is pointed along the direction of the scattered radiation and the two other basis vectors lie in the perpendicular observation plane. The incident unpolarized radiation comes into the formulae as components of the stress-energy tensor with respect to this reference frame. Our results are illustrated on a simple example of a cloud illuminated by a single beam of radiation.

1 Introduction

Electron scattering is an important phenomenon that influences observed radiation as well as dynamics of electrons in different types of astrophysical objects. Between 'classical' systems of interest are relativistic jets in active galactic nuclei. Many authors have studied an effect of Thomson scattering on jet velocity profiles (see e.g. Noerdlinger 1974; O'Dell 1981; Sikora & Wilson 1981 and Phinney 1982 for pioneering papers). The deceleration by ambient radiation field (also called radiation drag) has been recognized as an important factor that determines terminal speeds of jets (Sikora et al. 1996; Fukue 2005). These ideas were more recently reconsidered in a connection with jets emerging from several galactic X-ray binaries (e.g. Renaud & Henri 1998).

In some sources, light scaterred by fast moving jets represent an important component of observed radiation. The scattering on relativistic electrons increases the photon energy up to X-rays or γ-rays and contributes to a non-zero linear polarization. Because of a small optical depth $\tau \ll 1$, one usually considers only single scattering. One of the first calculations of the polarization due to Thomson scattering in blazars was given by Begelman & Sikora (1987). The relative importance of the synchrotron and Compton scattered radiation in the polarized radiation from blazars was discussed by Poutanen (1994). The polarimetry may be important also in the case of radiation from gamma-ray bursts since it can help to discriminate between various geometries of their sources (Lazzati et al. 2004).

Here, we consider intensity and polarization of light scattered on a hot optically thin cloud. The scatterers are relativistic electrons moving randomly in the cloud reference frame. This

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problem was extensively studied by Nagirner & Poutanen (1993), who considered scattering in an isotropic electron gas with Maxwellian or power-law energy distribution. Their rigorous formalism uses Klein-Nishina cross-section formula and allows to consider highly energetic as well as polarized incident radiation. On the other hand, in some astrophysical applications single Thomson scattering provide sufficient approximation. The main results of the work presented here are three very simple formulae for the *frequency-integrated* Stokes parameters for the scattered radiation. The incident radiation field is included in terms of the relativistic stress-energy tensor $T^{\mu\nu}$. The derivation of the formulae is described in Sec. 2. Sec. 3 is devoted to a particular example of scattering on a cloud with monoenergetic electron distribution. Discussion and some concluding remarks follow in Sec. 4.

2 Stokes parameters of the scattered light

We consider Thomson scattering on electrons of warm cloud. The cloud is assumed to be optically thin so that it is sufficient to consider only single scattering. The Compton optical depth is $\tau = n\,\sigma_{\mathrm{T}} R$, where n is the electron number density, σ_{T} is the total Thomson cross-section and R is the size of the cloud. Electrons are moving with random velocity and their distribution is isotropic in the rest frame of the cloud (referred to as CF). The frame is defined by the orthonormal tetrad $\{\boldsymbol{u}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}\}$, where the time-like four-vector $\boldsymbol{u}$ denotes cloud four-velocity. The observer is located along the $\boldsymbol{Z}$-vector that is introduced as a spatial projection of the four-momentum $\boldsymbol{p}$ of scattered photons, $\boldsymbol{Z} = \boldsymbol{u} - \boldsymbol{p}/\nu$, where $\nu = -\boldsymbol{p}\cdot\boldsymbol{u}$ is the frequency of the scattered radiation in CF. The remaining two four-vectors, $\boldsymbol{X}$ and $\boldsymbol{Y}$ can be chosen arbitrary in the calculations. They form the polarization basis with respect to which the Stokes parameters of the scattered radiation are calculated. Spatial components of vectors and tensors with respect to CF are denoted by capital letters and indices are raised/lowered using the special-relativistic metric tensor $\eta_{\alpha\beta} = \mathrm{diag}(-1, 1, 1, 1)$.

The Lorentz factor γ and the relative velocity $\boldsymbol{\beta}$ of an electron with a four-velocity $\tilde{\boldsymbol{u}}$ measured by an observer in CF are given by $\gamma = -\tilde{\boldsymbol{u}}\cdot\boldsymbol{u}$ and $\boldsymbol{\beta} = \boldsymbol{u} - \tilde{\boldsymbol{u}}/\gamma$. Because $\boldsymbol{\beta}$ is a normalized projection of $\tilde{\boldsymbol{u}}$ onto the three-space perpendicular to $\boldsymbol{u}$, its time component vanishes in CF. The non-vanishing components can be expressed as

$$\beta^X = \beta\,\sin\theta\,\cos\phi, \quad \beta^Y = \beta\,\sin\theta\,\sin\phi, \quad \beta^Z = \beta\,\cos\theta, \tag{1}$$

where θ and ϕ are referred to as the polar and azimuthal angle. One can easily check that $\gamma = (1-\beta^2)^{-1/2}$, where $\beta^2 \equiv \boldsymbol{\beta}\cdot\boldsymbol{\beta}$.

The isotropic electron distribution in the cloud frame is described by the electron distribution function $n(\boldsymbol{\beta}) = nf(\gamma)$, where n is the electron number density and the function $f(\gamma)$ is normalized to unity. Due to their additivity, the total Stokes parameters[1] $S = I, Q, U$ can be expressed as an integral

$$S = \int_\gamma f(\gamma) \int_\phi \int_\theta s(\gamma, \phi, \theta)\,\mathrm{d}\gamma\,\mathrm{d}\phi\,\mathrm{d}\theta, \tag{2}$$

where $s(\gamma, \phi, \theta)$ are Stokes parameters ($s = i, q, u$) of the radiation scattered by a swarm of electrons moving with the Lorentz factors from the interval $\langle\gamma, \gamma + \mathrm{d}\gamma\rangle$ in the direction

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[1] We remove the circularity V-parameter from our discussion because the Thomson scattering produces strictly linear polarization for which $V = 0$.

described by the azimuthal and polar angles in the intervals $\langle\phi, \phi + \mathrm{d}\phi\rangle$ and $\langle\theta, \theta + \mathrm{d}\theta\rangle$. The Stokes parameters are measured in CF and the swarm has electron number density n (the same as the density of the whole cloudlet).

For the swarm we introduce a comoving frame with the orthonormal tetrad $\{\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}}, \tilde{\boldsymbol{Z}}\}$ [referred to as swarm frame (SF)], where the time-like four-vector $\tilde{\boldsymbol{u}}$ is the electron four-velocity and $\tilde{\boldsymbol{Z}}$ is projected four-momentum $\boldsymbol{p}$ onto the three-space perpendicular to $\tilde{\boldsymbol{u}}$, $\tilde{\boldsymbol{Z}} = \tilde{\boldsymbol{u}} - \boldsymbol{p}/\tilde{\nu}$. The frequency of the scattered radiation measured in SF is $\tilde{\nu} = -\boldsymbol{p}\cdot\tilde{\boldsymbol{u}}$. In addition, we chose the $\tilde{\boldsymbol{Y}}$-vector so that it is perpendicular to all three fourvectors $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}$, $\boldsymbol{p}$.

The Stokes parameters of the scattered radiation $\tilde{\imath}$, $\tilde{q}$, $\tilde{u}$ with respect to SF can be expressed in terms of the stress-energy tensor of the incident radiation field as (Sobolev 1963, Beloborodov 1998, see also Horák & Karas 2005)

$$\tilde{\imath} = A\left(\tilde{T}^{tt} + \tilde{T}^{ZZ}\right), \quad \tilde{q} = A\left(\tilde{T}^{YY} - \tilde{T}^{XX}\right), \quad \tilde{u} = A\left(\tilde{T}^{XY} + \tilde{T}^{YX}\right), \tag{3}$$

where $A \equiv 3\tau/16\pi$ and $\tilde{T}^{\mu\nu}$ denotes components of the stress-energy tensor in SF. They can be expressed in terms of the CF-components $T^{\mu\nu}$ using a Lorentz transform $\Lambda^{\mu}_{\nu}(\gamma, \theta, \phi)$ between CF and SF. With the aid of the transformation, equations (3) can be uniformly written as

$$\tilde{s} = A\,\tilde{M}^{(s)}_{\rho\sigma}\,T^{\rho\sigma} \tag{4}$$

with

$$\tilde{M}^{(i)}_{\rho\sigma} = \Lambda^{t}_{\rho}\Lambda^{t}_{\sigma} + \Lambda^{Z}_{\rho}\Lambda^{Z}_{\sigma}, \quad \tilde{M}^{(q)}_{\rho\sigma} = \Lambda^{Y}_{\rho}\Lambda^{Y}_{\sigma} - \Lambda^{X}_{\rho}\Lambda^{X}_{\sigma}, \quad \tilde{M}^{(u)}_{\rho\sigma} = \Lambda^{X}_{\rho}\Lambda^{Y}_{\sigma} + \Lambda^{Y}_{\rho}\Lambda^{X}_{\sigma}. \tag{5}$$

Let us consider a special case when the relative velocity of the electron swarm $\boldsymbol{\beta}$ lies in the $\boldsymbol{X} - \boldsymbol{Z}$ plane in CF (the azimuthal angle $\phi = 0$). The $\boldsymbol{Y}$-vector is perpendicular to all three four-vectors $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}$ and $\boldsymbol{p}$ so that the Y-axes of the both reference frames are aligned, $\tilde{\boldsymbol{Y}} = \boldsymbol{Y}$. In that case all three Stokes parameters are transformed in the same way as the integrated intensity, keeping the fraction s/ν^4 invariant (Cocke & Holm 1972). In a general case, when the velocity $\boldsymbol{\beta}$ has non-zero Y-component in CF, we perform a rotation of the frame about the Z-axis by angle ϕ. Applying the well known transformation rules of Stokes parameters under rotations (e.g. Rybicky & Lightman 1979), we find that

$$i = \delta^4\tilde{\imath}, \quad q = \delta^4\left(\tilde{q}\cos 2\phi - \tilde{u}\sin 2\phi\right), \quad u = \delta^4\left(\tilde{q}\sin 2\phi + \tilde{u}\cos 2\phi\right), \tag{6}$$

where $\delta \equiv \nu/\tilde{\nu} = [\gamma(1 - \beta\cos\theta)]^{-1}$ is Doppler factor.

The above transformations are valid if the source moves *as a whole* with respect to the observer. In our case, however, the source is *stationary* in the sense that it contains electrons with fast individual motions in random directions and a direction of motion of an individual electron is frequently changed. Hence photons are essentially radiated from the constant place. In the former case, the Lorentz transformation contains also a contribution of the aberration effect. The Stokes parameters are expressed per time of observation $\mathrm{d}t$ in CF and per time of emission $\mathrm{d}\tilde{t}$ in SF. These two time intervals are related by $\mathrm{d}\tilde{t} = \gamma\mathrm{d}t$ if the source is stationary and radiates essentially from the same point or by $\mathrm{d}\tilde{t} = \gamma(1 - \beta\cos\theta)\mathrm{d}t$ if the source is in a bulk motion and its distance from the observer is changing as $ct(1-\beta\cos\theta)$. For this reason, we should complete the Lorentz transformation by an extra factor $(1-\beta\cos\theta) = 1/(\gamma\delta)$ (see also Begelman & Sikora 1987, Blumenthal & Gould 1970 and Rybicky & Lightman 1979 sec. 4.8).

The expressions for the transformed Stokes parameters i, q and u can be written as

$$s = A\, M^{(s)}_{\rho\sigma}\, T^{\rho\sigma} \tag{7}$$

with matrices $\mathsf{M}^{(s)}$ defined as

$$\begin{aligned}
\mathsf{M}^{(i)} &= \frac{\delta^3}{\gamma}\,\tilde{\mathsf{M}}^{(i)}, && (8)\\
\mathsf{M}^{(q)} &= \frac{\delta^3}{\gamma}\left(\tilde{\mathsf{M}}^{(q)}\cos 2\phi - \tilde{\mathsf{M}}^{(u)}\sin 2\phi\right), && (9)\\
\mathsf{M}^{(u)} &= \frac{\delta^3}{\gamma}\left(\tilde{\mathsf{M}}^{(q)}\sin 2\phi + \tilde{\mathsf{M}}^{(u)}\cos 2\phi\right), && (10)
\end{aligned}$$

and the matrices $\tilde{\mathsf{M}}^{(s)}$ defined in equation (5).

The total Stokes parameters I, Q, U can be obtained by substituting equation (7) into the expression (2). The dependence of the Stokes parameters s on the Lorentz factor γ and direction of motion (angles ϕ and θ) is hidden in the matrix $\mathsf{M}^{(s)}$ in equation (7). Hence, the stress-energy tensor of the incident radiation can be put outside the integral and knowing the distribution function $f(\gamma)$ the rest can be integrated. The Lorentz transformation $\Lambda^{\alpha}_{\beta}(\gamma,\theta,\phi)$ is derived in the Appendix.

By integrating $\mathsf{M}^{(i)}$, we find

$$\int_\gamma\int_\phi\int_\theta f\,\mathsf{M}^{(i)}\mathrm{d}\theta\,\mathrm{d}\phi\,\mathrm{d}\gamma = \begin{pmatrix} 1+\mathcal{A} & 0 & 0 & -\mathcal{A}\\ 0 & \mathcal{B} & 0 & 0\\ 0 & 0 & \mathcal{B} & 0\\ -\mathcal{A} & 0 & 0 & 1+\mathcal{A}-2\mathcal{B}\end{pmatrix}, \tag{11}$$

where we define quantities

$$\mathcal{A} \equiv \tfrac{4}{3}\left\langle \gamma^2\beta^2\right\rangle, \quad \mathcal{B} \equiv 1 - \left\langle \frac{\ln[\gamma(1+\beta)]}{\beta\gamma^2}\right\rangle, \tag{12}$$

with notation $\langle\, x\,\rangle \equiv \int x\, f(\gamma)\mathrm{d}\gamma$ for averaging over the electron Lorentz factor γ. According to equation (2), the total intensity scattered into the direction $\boldsymbol{Z}$ in CF can be expressed as

$$I = A\left[(1+\mathcal{A})\left(T^{tt}+T^{ZZ}\right) + \mathcal{B}\left(T^{tt}-3T^{ZZ}\right) - 2\mathcal{A}T^{tZ}\right], \tag{13}$$

where we use the identity $T^{XX}+T^{YY} = T^{tt}-T^{ZZ}$.

Similar calculations lead to the formulae for two other Stokes parameters,

$$Q = A\left(T^{YY}-T^{XX}\right) \quad \text{and} \quad U = -2\,A\,T^{XY}. \tag{14}$$

These are, however, same as if the radiation was scattered by a cold cloud.

3 Example: A monoenergetic electron distribution

We illustrate our results on a simple example. The electron distribution function in CF is monoenergetic, $f(\gamma) = \delta(\gamma-\gamma_0)$. All electrons have the same energy $\gamma_0 m_{\mathrm{e}}$, with m_{e} being the electron rest mass. The averaging in expressions (12) is trivial. We obtain

$$\mathcal{A} = \tfrac{4}{3}(\gamma_0^2 - 1), \quad \mathcal{B} = 1 - \frac{\ln\left[\gamma_0 + \sqrt{\gamma_0^2 - 1}\right]}{\gamma_0\sqrt{\gamma_0^2 - 1}}. \tag{15}$$

We consider a narrow beam of incident radiation that propagates in the direction $\boldsymbol{n}$ in the cloud reference frame[2]. This four-vector and the direction of observation $\boldsymbol{Z}$ make the angle ϑ. Obviously, all properties of the scattered radiation depend only on this angle and we can assume without any loss of generality that $n^Y = 0$. The two other components are $n^X = \sin\vartheta$ and $n^Z = \cos\vartheta$. The integrated intensity of the incident radiation is I_0 and the nonzero components of the stress-energy tensor of the incident radiation are

$$T^{tt} = I_0, \quad T^{tZ} = I_0\, n^Z, \quad T^{ZZ} = I_0\, n^Z n^Z, \quad T^{XX} = I_0\, n^X n^X. \tag{16}$$

The total amount of radiation scattered on the cloud is proportional to τI_0. We use it to introduce normalized Stokes parameters $I_\star$ and $Q_\star$ of the scattered radiation. The remaining parameter U is zero because of the symmetry with respect to the $X - Z$ plane. Using equations (13), (14) and (16) we find

$$I_\star \equiv \frac{I}{\tau I_0} = \frac{3}{16\pi}\left[(1 + \mathcal{A} - 3\mathcal{B})\cos^2\vartheta - 2\mathcal{A}\cos\vartheta + (1 + \mathcal{A} + \mathcal{B})\right], \tag{17}$$

$$Q_\star \equiv \frac{Q}{\tau I_0} = -\frac{3}{16\pi}\sin^2\vartheta. \tag{18}$$

The angular dependence of the scattered intensity $I_\star$ on the angle of observation ϑ for different values of γ_0 is shown in the left panel of Figure 1. In the case of a cloud containing cold electrons ($\gamma_0 = 1$) both $\mathcal{A}$ and $\mathcal{B}$ are zero and the angular dependence of the scattered intensity reduces to $I \propto 1 + \cos^2\vartheta$ – the same as for Thomson scattering on a single electron. The scattering on a cold cloud is symmetric with respect to the plane perpendicular to the direction of the incident radiation beam. On the other hand, if the cloud contains ultrarelativistic electrons with $\gamma_0 \gg 1$, we have $\mathcal{A} \approx 4/3\gamma_0^2$, $\mathcal{B} \approx 1$ and equation (13) gives the angular dependence $I \propto \gamma_0^2(1 - \cos\vartheta)^2$, which is highly asymmetric. Most of the radiation is scattered in the backward direction. This is important regarding a dynamics of the cloud: the scattered radiation transports momentum from the electrons in the backward direction so that the hot clouds are more strongly accelerated by the incident radiation. This effect called *Compton rocket*, was firstly studied by O'Dell (1981) and later reconsidered by Phinney (1982).

The direction of polarization of the scattered radiation is parallel to the $\boldsymbol{Y}$-vector in CF because $Q_\star < 0$. (see a discussion of the polarization direction in Horák & Karas 2005). The polarization magnitude is given as a ratio $\Pi = |Q_\star|/I_\star$, because both U and V vanish. The angular dependence of the polarization magnitude is shown in the right panel of Figure 1. In the case of the cold cloud, the expression for the polarization is identical with the well known expression for the scattering on a single electron, $\Pi = (1 - \cos^2\vartheta)/(1 + \cos^2\vartheta)$ giving the maximal value of unity for a completely polarized radiation when $\vartheta_{\mathrm{m,cold}} = \pi/2$. However,

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[2] The fourvector $\boldsymbol{n}$ is a projection of the incident photon four-momentum onto the three-space perpendicular to the four-velocity $\boldsymbol{u}$.

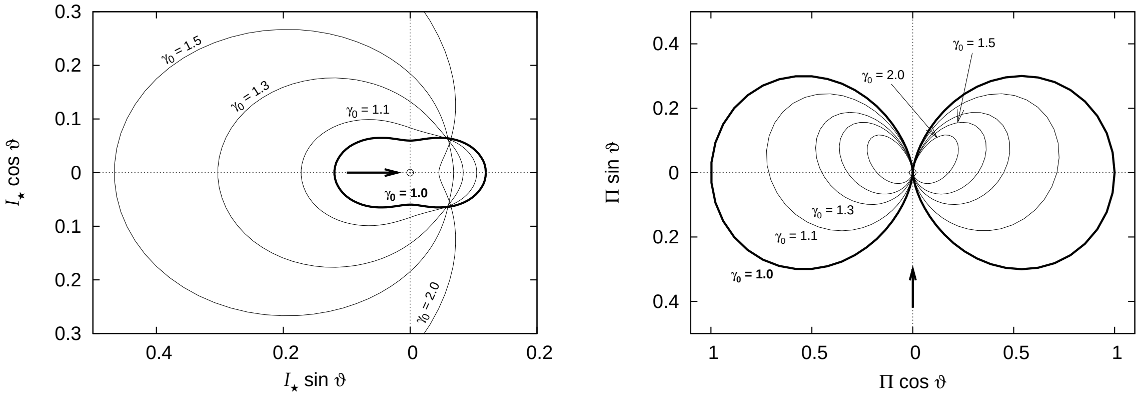

Figure 1: Left: the normalized scattered intensity $I_\star$ as a function of the scattering angle ϑ between the direction of the incident radiation beam and the direction of observation. The scattering occurs on the hot electron cloud. Different curves corresponds to different values of the electron Lorentz factor measured in the blob reference frame. The case of cold electron corresponds to $\gamma_0 = 1$. Right: the magnitude of transversal polarization Π as a function of the scattering angle for several values of the electron Lorentz factor. The depolarization effect of the electron motions and the shift of the angle of maximal polarization are apparent.

the polarization is reduced by a factor $\sim \gamma_0^2$ if the cloud contains relativistic electrons. The maximal polarization occurs closer to the direction of the incident radiation, because of the asymmetric profile of the scattered intensity. Simple algebra gives the angle ϑ_m along which an observer receives radiation with the highest polarization

$$\cos\vartheta_\mathrm{m} = \frac{1}{\mathcal{A}}\left[1 + \mathcal{A} - \mathcal{B} - \sqrt{(1-\mathcal{B})(1+2\mathcal{A}-\mathcal{B})}\right]. \tag{19}$$

This angle approaches zero as the electron Lorentz factor increases. However, the polarization is strongly reduced in that case.

4 Conclusions

In this note we studied polarization properties of the Thomson-scattered radiation on the cloud with relativistic electrons. The freqeuncy integrated Stokes parameters are given by equations (13) and (14). The incident unpolarized radiation appears comes into the formulae as components $T^{\alpha\beta}$ of the stress-energy tensor with respect to the comoving polarization frame. The same quantity determines also a dynamics of the cloud (see e.g. Phinney 1982). Our results are useful when one considers the polarization effect on the scattered radiation together with the dynamical effects on the cloud (see e.g. Horák & Karas 2005).

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Appendix: Lorentz transform

Here, we derive the Lorentz transform $\Lambda^{\alpha}_{\beta}(\theta,\phi)$ between CF and SF. First, let us consider the case $\phi = 0$. The Y-axes of both frames are aligned ($\boldsymbol{Y} = \tilde{\boldsymbol{Y}}$) as it was discussed in section 2. The remaining tetrad four-vectors $\boldsymbol{X}$, $\tilde{\boldsymbol{X}}$, $\boldsymbol{Z}$ and $\tilde{\boldsymbol{Z}}$ of both frames can be expressed as linear combinations of $\boldsymbol{u}$, $\tilde{\boldsymbol{u}}$ and $\boldsymbol{p}$. The components of the Lorentz transform matrix can be expressed as scalar products

$$\Lambda^{\alpha}_{\beta} = \tilde{\boldsymbol{e}}^{(\alpha)} \cdot \boldsymbol{e}_{(\beta)}, \tag{20}$$

of corresponding covariant and contravariant basis four-vectors $\tilde{\boldsymbol{e}}^{(\alpha)} = \{\tilde{\boldsymbol{u}}, \tilde{\boldsymbol{X}}, \tilde{\boldsymbol{Y}}, \tilde{\boldsymbol{Z}}\}$ and $\boldsymbol{e}_{(\alpha)} = \{-\boldsymbol{u}, \boldsymbol{X}, \boldsymbol{Y}, \boldsymbol{Z}\}$, respectively. We find

$$\Lambda^{\alpha}_{\beta}(\theta) = \begin{pmatrix} \gamma & -\gamma\beta\sin\theta & 0 & -\gamma\beta\cos\theta \\ -k & 1 & 0 & k \\ 0 & 0 & 1 & 0 \\ l & -\gamma\beta\sin\theta & 0 & m \end{pmatrix} \tag{21}$$

with

$$k = \gamma\delta\beta\sin\theta, \quad l = \gamma^2\beta\delta(\beta - \cos\theta), \quad m = \delta - \gamma\beta\cos\theta. \tag{22}$$

In the more general case when the three-velocity $\vec{\beta}$ and $\boldsymbol{Y}$-vector make nonzero azimuthal angle ϕ, we make the rotation about the $\boldsymbol{Z}$-axis through angle ϕ. The final Lorentz transform is given by $\mathsf{\Lambda}(\theta,\phi) = \mathsf{\Lambda}(\theta)\mathsf{R}_Z(\phi)$. We find

$$\Lambda^{\alpha}_{\beta}(\phi,\theta) = \begin{pmatrix} \gamma & -\gamma\beta\sin\theta\cos\phi & -\gamma\beta\sin\theta\sin\phi & -\gamma\beta\cos\theta \\ -k & \cos\phi & \sin\phi & k \\ 0 & -\sin\phi & \cos\phi & 0 \\ l & -\gamma\beta\sin\theta\cos\phi & -\gamma\beta\sin\theta\sin\phi & m \end{pmatrix}. \tag{23}$$