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Disk instability as an energy source for quasi-periodic oscillations

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ABSTRACT
We discuss a scenario, in which the energy for quasi-periodic oscillations comes from instabilities of the accretion disk. We demonstrate this mechanism on the interaction among stable and unstable modes of oscillations in slender accretion tori. In our model, the unstable non-axisymmetric corotation modes are nonlinearly coupled to stable acoustic modes of torus oscillations. The increasing energy of corotation modes is advected to the acoustic modes by the means of parametric instability. The stable modes may reach substantial amplitudes even if they are damped according to the linear theory.

Keywords: black hole physics – accretion disks – instabilities

1 INTRODUCTION

It has been proposed that a nonlinear resonance among two accretion disk oscillations is responsible for observed quasiperiodic oscillations (QPOs) in both black-hole and neutron-star sources (see Kluźniak 2005 for a review). In earlier versions of the resonance model, the two QPOs were connected to the orbital and radial epicyclic motion. More recently, QPOs has been interpreted as radial and vertical epicyclic oscillations of the accreted matter at the position of the 3:2 parametric resonance (Kluzniak and Abramowicz, 2002).

The source of energy for the oscillations has not been identified yet. In the parametric-resonance models the vertical oscillations are excited and fed by the radial oscillations. If the feedback of vertical to radial oscillations is taken into account, the two modes periodically exchange the energy keeping the total energy constant. In order to solve this problem, two possible sources of energy has been proposed: (1) external periodic forcing e.g. by spinning central neutron star and (2) internal stochastic forcing by hydrodynamic or magnetohydrodynamic turbulence. The former process was proposed by Lee et al. (2004) and was largely motivated by an observation that in the two millisecond pulsars showing QPOs the QPO frequency difference is always close to the observed spin frequency or half of it (Wijnands et al., 2003; Linares et al., 2005). The latter was recently examined by Brandenburg (2005) and Arras et al. (2006) in the shearing-box simulations and by Vio et al. (2006) in the simplified model of stochastic excitation of test-particle epicyclic motion.

In this work we propose an alternative mechanism in addition to these two scenarios. In our view the QPOs are fed by a linear instability of the accretion disk through a nonlinear modal coupling. As an example of this process we start to examine nonlinear interactions among stable and unstable modes of slender accretion tori. The source of the energy is Papaloizou-Pringle instability. The plan of the paper is the following. Growth rates and eigen-functions of the unstable modes are summarized in Sec. 2. The mechanism of nonlinear interactions among stable and unstable modes is outlined in Sec. 3. In Sec. 4 we show that the two necessary conditions for these interactions (a particular combination of eigenfrequencies of the modes and non-zero coupling coefficient) are satisfied in slender tori. Finally, Sec. 5 is devoted to a discussion and conclusions.

2 LINEAR INSTABILITY OF SLENDER TORI

An important class of torus oscillations is represented by corotation modes, for which the mode pattern corotates with the fluid at a particular radius inside the torus (corotation radius). According to classical works in the subject (e.g. Goldreich et al., 1986), these modes are suspected to be unstable. Indeed, in the limit of infinitely slender tori these modes are just marginally stable. They are described by eigenfunctions and eigenfrequencies of the form

$$W_0 = C_0 \mathrm{e}^{\mathrm{i} m_0 \phi}, \quad w_0 = m_0 \Omega_0, \tag{1}$$

where C_0 is a normalization constant, m_0 is an integer azimuthal wavenumber and Ω_0 is the flow angular velocity at the maximal pressure radius that coincides with the local Keplerian frequency, $\Omega_0 = \Omega_{\mathrm{K}}(r_0)$. The perturbation quantity W is connected to the Eulerian pressure perturbation by $\delta p = (w - m_0\Omega)\rho W$. The slender torus corresponds to the limit $\beta \sim \Delta r/r \to 0$ (Δr is the radial size of the torus).

The modal eigenfrequencies and eigenfunctions of larger tori can be found using a perturbative expansion in β. (Blaes, 1985; Blaes et al., 2007). For tori with constant angular momentum distribution this procedure gives

$$w_0 = m_0 \Omega_0 + \mathrm{i}\sqrt{2}\, m_0\, b\, \beta + \mathcal{O}(\beta^2), \tag{2}$$

and

$$W_0 = C_0 \mathrm{e}^{\mathrm{i} m_0 \phi} \left\{ 1 + m_0^2 \beta^2 \left[a^2 \bar{x}^2 - b^2 \bar{y}^2 \frac{4\sqrt{2}\,\mathrm{i}b}{\bar{\varpi}_r^2} \bar{x} + \frac{\bar{\varpi}_r^2 b^2 - \bar{\varpi}_z^2 a^2}{2(n+1)\bar{\varpi}_r^2 \bar{\varpi}_z^2} \right] + \mathcal{O}(\beta^3) \right\}, \tag{3}$$

where the coefficients a and b are given by

$$a^2 \equiv \frac{4(1+2n) + \bar{\varpi}_r^2}{4(1+n)\bar{\varpi}_r^2}, \quad b^2 \equiv \frac{4 - \bar{\varpi}_r^2}{4(1+n)\bar{\varpi}_r^2}. \tag{4}$$

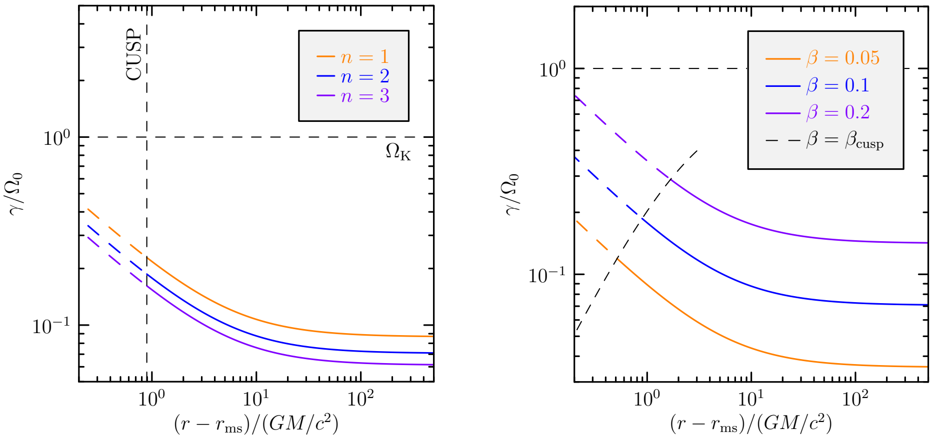

Figure 1. The growth-rate of the Papaloizou-Pringle instability normalized by the local Keplerian angular velocity Ω_0 as function of the location of the torus center for different values of the polytropic index (*left*) and of the torus thickness β (*right*). The limiting cases of the torus with cusp are denoted by the dashed line.

The eigenfunctions are expressed using 'slender' coordinates $\bar{x}$ and $\bar{y}$ contracting with the torus as $\beta \to 0$ (see Blaes et al. 2006 for a definition). The corotation mode of the slender torus splits into two modes. The positive or negative values of b give the unstable or stable mode, respectively, where the unstable mode corresponds to the principal mode of the Papaloizou-Pringle instability (Blaes, 1985).

The external gravitational potential comes into the above formulae only through the normalized radial and vertical epicyclic frequencies $\bar{\omega}_r = \omega_r/\Omega_0$ and $\bar{\omega}_z = \omega_z/\Omega_0$. In the spherically symmetric Newtonian gravitation field both epicyclic frequencies equal to the Keplerian orbital frequency, $\bar{\omega}_r = \bar{\omega}_z = 1$ and we recover equations (4.11) and (4.13) of Blaes (1985).

The growth-rates of the instability (given by the imaginary part of w_0) as functions of the location of the torus are shown in Fig. 1 for different values of the polytropic index n and the thickness of the torus β. In this case, we consider the pseudo-Newtonian potential (Paczynsky and Wiita, 1980) $\Phi = GM/(r - r_\mathrm{s})$ with $r_\mathrm{s} \equiv 2GM/c^2$ being the Schwarzschild gravitational radius. The instability is stronger when the torus approaches the location of the marginally stable circular orbit r_ms. For a given value of β, however, there exists a limiting radius bellow which the matter in the torus starts to accrete onto the black hole and the assumption about stationary equilibrium breaks down.

Our results have been obtained for constant angular momentum tori that are violently unstable. The instability is largely reduced for steep angular momentum distributions (e.g. if $q > \sqrt{3}$, where the flow angular velocity is parameterized as $\Omega \propto r^{-q}$).

3 INTERACTIONS AMONG STABLE AND UNSTABLE MODES

When the amplitude of the corotation mode is sufficiently large, nonlinear processes become important. In principle, growth of the instability can be even halted by nonlinear interaction with other oscillation modes, if they are damped significantly. The energy of the unstable mode is advected to the damped modes, where it is dissipated. This process likely plays a key role in limiting amplitudes of the pulsations of ZZ Ceti stars, dwarf-type variables and some δ Scuti stars (Dziembowski, 1982; Nowakowski, 2005; and references therein). It seems to be also important for the saturation of the r-mode instability in neutron stars (Arras et al., 2003).

The mechanism of this process is as follows. Let us consider two stable damped modes ('daughter' modes) whose eigenfrequencies and excitation rates are ω_1, ω_2 and γ_1, $\gamma_2 < 0$ (the complex eigenfrequencies are given by $w_i = \omega_i + \mathrm{i}\gamma_i$). They will form a resonant triple with the unstable ('parent') mode with frequency ω_0 and excitation rate $\gamma_0 > 0$ if the condition of a combination resonance,

$$\omega_1 + \omega_2 + \omega_0 \equiv \delta\omega \approx 0, \tag{5}$$

is satisfied. Neglecting any influence of the other modes, the oscillations can be described by the Lagrangian displacement

$$\boldsymbol{\xi}(t, \boldsymbol{x}) = \Re\left[A_0(t)\,\mathrm{e}^{-\mathrm{i}\omega_0 t}\,\boldsymbol{\xi}_0(\boldsymbol{x}) + A_1(t)\,\mathrm{e}^{-\mathrm{i}\omega_1 t}\,\boldsymbol{\xi}_1(\boldsymbol{x}) + A_2(t)\,\mathrm{e}^{-\mathrm{i}\omega_2 t}\,\boldsymbol{\xi}_2(\boldsymbol{x})\right], \tag{6}$$

where $\boldsymbol{\xi}_i(\boldsymbol{x})$ are the Lagrangian eigenfunctions of the interacting modes. The resonance causes a slow modulation of the phases ϕ_i and amplitudes a_i of the oscillations that can be described by slowly varying dimensionless complex amplitudes $A_i = a_i \mathrm{e}^{\mathrm{i}\phi_i}$. Their time-behavior is given by the amplitude equations of the form

$$\dot{A}_1 = \gamma_1 A_1 + \mathrm{i}\,\omega_1 \kappa^* A_2^* A_0^* \mathrm{e}^{\mathrm{i}\delta\omega t}, \tag{7}$$
$$\dot{A}_2 = \gamma_2 A_2 + \mathrm{i}\,\omega_2 \kappa^* A_1^* A_0^* \mathrm{e}^{\mathrm{i}\delta\omega t}, \tag{8}$$
$$\dot{A}_0 = \gamma_0 A_0 + \mathrm{i}\,\omega_0 \kappa^* A_1^* A_2^* \mathrm{e}^{\mathrm{i}\delta\omega t}, \tag{9}$$

where κ is the three-mode coupling coefficient determined by the eigenfunctions of the modes.

The generic behavior of the amplitudes is illustrated by three examples in Fig. 2. The solutions are obtained by numerical integration of Eqs. (7)–(9) for different values of the parameters γ_i and $\delta\omega$. Initially, they behave according to the linear theory – the amplitude of the parent mode grows exponentially while the daughter modes are exponentially damped. Nonlinear effects become important after the amplitude A_0 overcomes certain threshold. The energy accumulated in the parent mode is then transferred to the daughter modes by means of the parametric instability. The threshold can be derived from Eqs. (7)–(9) assuming that the amplitudes A_1 and A_2 are much smaller than the amplitude of the parent mode, so that their influence on the parent mode can be neglected (Dziembowski, 1982; Nowakowski, 2005). In that case we may suppose that $A_0(t) \propto \mathrm{e}^{\gamma_0 t}$ and both $A_1(t)$ and $A_2(t)$

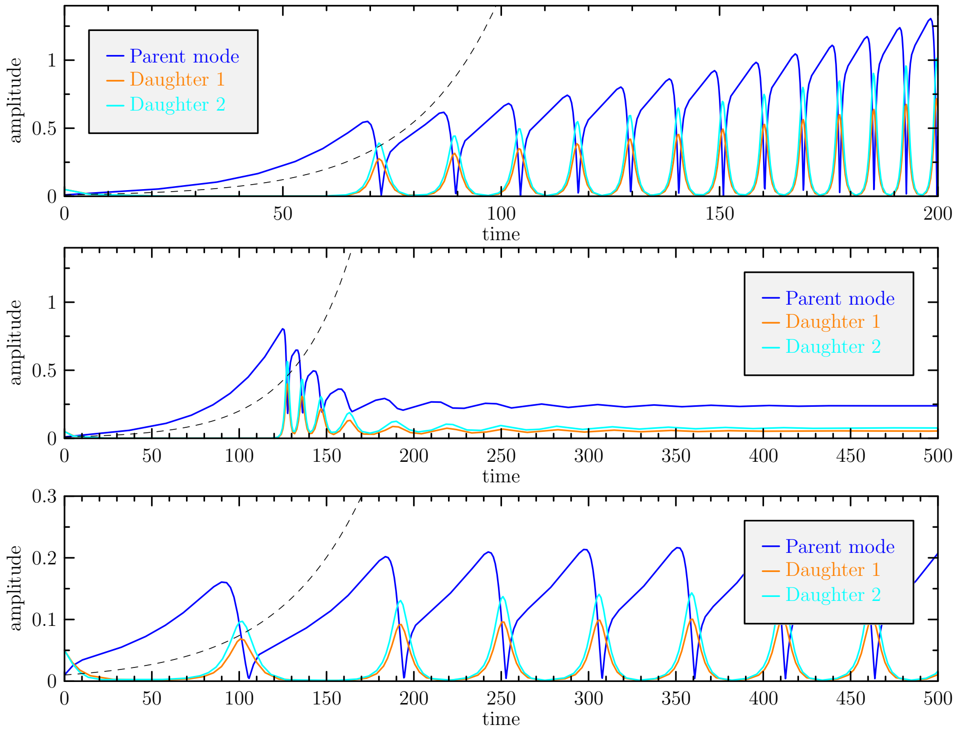

Figure 2. Evolution of the amplitudes in a resonant triple. While the parent mode is unstable, the two daughter modes are damped. Three types of possible evolution are shown: *Top:* The damping rates are too small compared to the growth-rate of the instability so that the daughter modes are not able to saturate the unstable parent mode. *Middle:* The energy from the instability is dissipated by the daughter modes and the system reach a steady state. *Bootm:* The accumulated energy of the parent mode is periodically advected to the daughter modes where it is dissipated.

are proportional to $\mathrm{e}^{\nu t}$. The real part of ν is positive when

$$A_0 \gtrsim A_{\mathrm{crit}} = \frac{1}{Q_1 Q_2\,|\kappa|^2}\left[1+\left(\frac{\delta\omega+\gamma_0}{\gamma_1+\gamma_2}\right)^2\right] \qquad (10)$$

with $Q_i \equiv \omega_i/\gamma_i$ being quality factors of the daughter modes.

Some fraction of the energy transferred from the parent mode is dissipated in the daughter modes, the rest of it flows back to the parent mode. Depending on the excitation rate γ_0, damping rates γ_1, γ_2 and on the frequency detuning $\delta\omega$, the whole process may lead to an unsaturated growth of all three modes, to a steady state or to periodic limit cycles. The unsaturated grow (Fig. 2a) occurs when $\gamma_1 + \gamma_2 \lesssim \gamma_0$, i.e. when the daughter modes are not able to dissipate the energy coming to the system from the instability. The instability is saturated by reaching the steady state (Fig. 2b) or by executing periodic limit cycles (Fig. 2c) when the total damping slightly overcome the excitation. The type of the saturation depends on the frequency detuning $\delta\omega$ (Wersinger et al., 1980; Dziembowski, 1982). Properties of the limit cycles in dependence of the system parameters were discussed in great details by Moskalik (1985). In particular, the period of the cycles is mostly influenced by the excitation rate $T \sim |\gamma_0|^{-1}$.

In practical situations, there can be several pairs of modes that satisfy the resonance condition (14). The resonant interactions among the unstable parent mode and N pairs of damped daughter modes have been recently studied by Nowakowski (2005). In this case, the necessary condition for the saturation is natural generalization of that for the single pair,

$$\sum_{i=1}^{N}(\gamma_{i,1} + \gamma_{i,2}) \geq \gamma_0, \tag{11}$$

where $\gamma_{i,1}$ and $\gamma_{i,2}$ are the damping rates of the daughter modes in i-th pair. When $N > 2$ it is impossible to reach the steady-state and the amplitudes are always strongly variable. For large N the amplitude of the parent mode suffers random changes and the system reach some kind of a statistical equilibrium.

4 PARAMETRIC INSTABILITY IN SLENDER TORI

Is the scenario described in the previous section relevant for the nonlinear behavior of the Papaloizou-Pringle instability of accretion tori? Essential is a presence of the daughter modes that are resonantly coupled to the unstable corotation mode. In this section we examine the necessary conditions for the existence of the coupled modes. The important question whether they are able to saturate the instability or not will be addressed in future.

Let us consider tori with $\beta = 0$. The eigenfunctions and eigenfrequencies of the non-axisymmetric modes can be expressed as

$$W_\alpha = \tilde{W}_\alpha\, \mathrm{e}^{\mathrm{i}m_\alpha\phi}, \quad \omega_\alpha = m_\alpha\Omega_0 + \tilde{\omega}_\alpha, \tag{12}$$

where $\tilde{W}_\alpha$ and $\tilde{\omega}_\alpha$ are the eigenfunction and eigenfrequency of the axisymmetric mode with $m_\alpha = 0$ (Blaes et al., 2006). The pattern speed of the mode is given by $\omega_{\mathrm{p},\alpha} = \omega_\alpha/m = \Omega_0 + \tilde{\omega}_\alpha/m_\alpha$. The daughter modes may be identified with modes whose patterns propagate with the same speed in the opposite direction with respect to the flow

$$\omega_1 = m_1\Omega_0 + \tilde{\omega}, \quad \omega_2 = m_2\Omega_0 - \tilde{\omega}. \tag{13}$$

The azimuthal wavenumbers m_1 and m_2 may be different. Combining the Eqs. (1) and (13), we find that

$$\delta\omega \equiv \omega_1 + \omega_2 + \omega_0 = 0 \quad \Leftrightarrow \quad m_1 + m_2 + m_0 = 0. \tag{14}$$

Our discussion may be extended to thicker tori by means of the perturbative expansion in β-parameter. The main effect is change of the eigenfrequencies due to the pressure corrections. These corrections are of the order of β^2 and depends also on the azimuthal wavenumber, because the pressure gradients in the azimuthal direction become important. The result is detuning of the perfect resonance whose magnitude is $\delta\omega \sim \beta^2$.

Next, we examine a coupling among these modes. The coupling coefficient for the interaction is given by the integral over the volume of the torus,

$$\kappa = \int_V f(\boldsymbol{\xi}_0, \boldsymbol{\xi}_1, \boldsymbol{\xi}_2) \mathrm{d}V, \tag{15}$$

where f is a multi-linear function in all its arguments. Except for eigenfunctions $\boldsymbol{\xi}_i$, f depends only on the quantities describing the equilibrium and there is no explicit dependence on the coordinates (Schenk et al., 2002). Because the equilibrium configuration is axially and equatorial-plane symmetric and the eigenfunctions depend on the azimuthal angle as $\boldsymbol{\xi}_\alpha \propto \exp[\mathrm{i} m_\alpha \phi]$, we find that the coupling coefficient may be nonzero only when

$$m_1 + m_2 + m_0 = 0. \tag{16}$$

We obtain the same selection rule as it is needed by the resonance condition (14), which implies that the resonant interaction is indeed possible. Finally, let us examine effects of two other symmetries. Because the corotation mode eigenfunction is even with respect to reflection and $\bar{y} \leftrightarrow -\bar{y}$ the daughter modes have to have the same parity with respect to this symmetry. Otherwise f is an odd function of z and the coupling coefficient vanishes.

5 DISCUSSION & CONCLUSIONS

According to the general discussion given in Sec. 3 stable oscillation modes of a fluid body may reach substantial amplitudes if they are nonlinearly coupled to a linearly unstable mode. In this work we have examined this process for the particular case of slender tori that are unstable with respect to the Papaloizou-Pringle instability. We have demonstrated that two necessary conditions for the resonance coupling: the frequency condition of a combination resonance and the selection rule imposed on the azimuthal wavenumbers are satisfied by many pairs of torus modes in the infinitely slender tori. A finite thickness of the torus cause a detuning of the perfect resonance which is of order of β^2. Careful analysis of the modal eigenfrequencies and eigenfunctions similar to that of Blaes et al. (2007) for epicyclic modes is needed in order to decide what modes are in the resonance in thicker tori.

The mechanism described above is unable to saturate the instability if the daughter modes are not sufficiently damped. This occurs in inviscid slender tori, where the instability leads to a fragmentation (Goodman et al., 1987). One possible source of damping may be a viscosity of the flow. Preliminary study shows that if the standard α-prescription is assumed the modes with significant pressure gradients are damped on the time-scale $t_{\mathrm{damp}} \sim (\alpha\Omega_0)^{-1}$ (the timescale for the epicyclic modes is comparable with that of the secular evolution of the torus $\beta^{-2}[\alpha\Omega_0]^{-1}$). Comparing it with the timescale of the instability growth, we realize that the corotation mode may be saturated if $\beta \lesssim \alpha$.

The other possible source of damping is accretion. It has been shown that the growth of the instability is reduced significantly because the reflection of the waves at the inner edge of the torus is not perfect (Blaes, 1987). Likely, the same effect causes damping of other oscillation modes.

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