

# Of NBOs and kHz QPOs: a Low-Frequency Modulation in Resonant Oscillations of Relativistic Accretion Disks

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## Abstract

The origin of quasi periodic modulations of flux in the kilohertz range (kHz QPOs), observed in low-mass X-ray binaries, is usually assumed to be physically distinct from that of the “normal branch oscillations” (NBOs) in the Z-sources. We show that a low-frequency modulation of the kHz QPOs is a natural consequence of the non-linear relativistic resonance suggested previously to explain the properties of the high-frequency twin peaks. The theoretical results discussed here are reminiscent of the 6 Hz variations of frequency and amplitude of the kHz QPOs reported by Yu, van der Klis, and Jonker (2001, ApJ, 559, L29).

**Key words:** accretion disks — QPOs — stars: neutron — X-rays: binaries

## 1. Introduction

Power density spectra of the X-ray flux of low-mass X-ray binaries have a rich phenomenology almost entirely lacking a physical explanation (van der Klis 1989, 2000). Twin kHz QPOs in neutron star sources, and their kHz counterpart in black-hole systems, have attracted the most attention because their frequency is comparable to the orbital frequencies in the innermost dynamically stable regions of the accretion flow (Kluźniak et al. 1990; van der Klis et al. 1996). Ad hoc models of QPOs have not withstood the test of time, but a more fundamental approach of studying the eigenmodes of accretion disks remains promising (Wagoner 1999; Kato 2001). It seems necessary to include non-linear effects to be in agreement with observations (Kluźniak, Abramowicz 2003). Here, we wish to point out that even the simplest non-linear equations of motion have a rich structure which is yet to be explored fully in the context of QPOs.

It is now recognized that non-linear coupling in the motion of accreting fluid may be responsible for some of the observed features of (kHz) QPOs in neutron-star and black-hole systems. It has been suggested that a non-linear resonance may be responsible for these twin QPOs (Kluźniak, Abramowicz 2001, 2003; Kato 2003). Among the reasons to believe that these highest frequencies reflect a non-linear resonance between two oscillation modes of a disk, probably occurring only in strong-field gravity, are the 3 : 2 ratio of the frequencies of the twin QPOs in black holes (Abramowicz, Kluźniak 2001; McClintock, Remillard 2004), and the sub-harmonic frequency difference between the two QPOs observed in the accreting 2.5 ms pulsar (Wijnands et al. 2003; Kluźniak et al. 2004; Lee et al. 2004). It has been found that a parametric-like resonance can be excited between the radial and vertical epicyclic frequencies in a simplified mathematical model of fluid motion in Einstein’s gravity, and that the ratio of the

two kHz frequencies can be made to match closely the one observed in Sco X-1 if the source is slightly off-resonance (Abramowicz et al. 2003; Rebusco 2004). In this paper, we employ the same model of twin kHz QPOs and we investigate its consequences for the time behavior of predicted frequencies and amplitudes. We lay down our description within the framework of an (internal) resonance model describing the oscillatory modes of a conservative system. Details of the adopted method and calculations can be found elsewhere (Horák 2004).

## 2. The Model

We solved a very general system of two coupled nonlinear oscillators described by the following governing equations near the resonance  $\omega_\theta : \omega_r \approx 3 : 2$ ,

$$\delta\ddot{r} + \omega_r^2 \delta r = \omega_r^2 f(\delta r, \delta\theta, \delta\dot{r}, \delta\dot{\theta}), \quad (1)$$

$$\delta\ddot{\theta} + \omega_\theta^2 \delta\theta = \omega_\theta^2 g(\delta r, \delta\theta, \delta\dot{r}, \delta\dot{\theta}), \quad (2)$$

under the condition that these two equations are invariant with respect to reflection of time, i.e., the Taylor expansion of functions  $f$  and  $g$  does not contain odd powers of the time derivatives of  $\delta r$  and  $\delta\theta$ . These equations may be taken to approximate two modes of an accretion disk with eigenfrequencies close to the radial and vertical epicyclic frequencies,  $\omega_r$  and  $\omega_\theta$ , respectively. The functions  $f$ ,  $g$  depend on the space-time metric, as well as on the properties of the fluid flow. Any particular mechanism should specify the physical meaning of these functions and fix their values, e.g. in terms of the accretion rate, resonance radius and other parameters. Such special models have been indeed proposed and they are encompassed by the scheme developed herein. For example, a special form of  $f$  and  $g$  was assumed in Abramowicz et al. (2003) and Rebusco (2004), based on the idea of parametric resonance, but here we keep the discussion general and use the

special solution for the purpose of an example and a numerical check, as described below. In fact, from the mathematical point of view, the free parameters of the model are the expansion coefficients through the fourth order of the functions  $f$  and  $g$ , which unambiguously define the solution.

We applied the method of multiple time scales (Nayfeh, Mook 1979) and looked for a solution in the form of a series,

$$\delta r(t) = \epsilon r_1 + \epsilon^2 r_2 + \epsilon^3 r_3 + \epsilon^4 r_4 + \mathcal{O}(\epsilon^5), \quad (3)$$

$$\delta \theta(t) = \epsilon \theta_1 + \epsilon^2 \theta_2 + \epsilon^3 \theta_3 + \epsilon^4 \theta_4 + \mathcal{O}(\epsilon^5). \quad (4)$$

This approach enables us to find a uniformly converging solution, which does not suffer from secularly growing terms. We introduced new independent variables  $T_k \equiv \epsilon^k t$  and derived the conditions for the elimination of secular terms in the “fast” variable  $T_0 = t$ , up to the fourth order in the expansion parameter  $\epsilon \ll 1$ . We restrict ourselves to the leading terms of the expansions, which take the form

$$\delta r(t) = A_r(t)e^{i\omega_r t} + \text{c.c.}, \quad (5)$$

$$\delta \theta(t) = A_\theta(t)e^{i\omega_\theta t} + \text{c.c.} \quad (6)$$

The solvability conditions describe the time evolution of complex amplitudes, which we further rewrite in the form

$$A_r \equiv \frac{1}{2} a_r e^{i\phi_r}, \quad A_\theta \equiv \frac{1}{2} a_\theta e^{i\phi_\theta}. \quad (7)$$

The amplitudes and phases therefore satisfy equations

$$\dot{a}_r = \frac{\alpha \omega_r}{16} a_r^2 a_\theta^2 \sin \gamma, \quad (8)$$

$$\dot{a}_\theta = -\frac{\beta \omega_\theta}{16} a_r^3 a_\theta \sin \gamma, \quad (9)$$

$$\dot{\phi}_r = -\frac{\omega_r}{2} [\kappa_r a_r^2 + \kappa_\theta a_\theta^2] - \frac{\alpha \omega_r}{16} a_r a_\theta^2 \cos \gamma, \quad (10)$$

$$\dot{\phi}_\theta = -\frac{\omega_\theta}{2} [\lambda_r a_r^2 + \lambda_\theta a_\theta^2] - \frac{\beta \omega_\theta}{16} a_r^3 \cos \gamma, \quad (11)$$

where  $\gamma \equiv 2\phi_\theta - 3\phi_r - \sigma t$  and  $\sigma = 3\omega_r - 2\omega_\theta$  are a phase function and a detuning parameter, and  $\alpha$ ,  $\beta$ ,  $\kappa_r$ ,  $\kappa_\theta$ ,  $\lambda_r$ , and  $\lambda_\theta$  are real constants describing our system. The detuning parameter  $\sigma$  describes small departures of the eigenfrequencies from sharp rational ratio. In this way, the method of multiple scales allows us to capture the amplitude–frequency interaction that is present in the oscillating solution.

Equations (10) and (11) provide frequency correction  $\Delta\omega_r = \dot{\phi}_r$  and  $\Delta\omega_\theta = \dot{\phi}_\theta$  to the eigenfrequencies  $\omega_r$  and  $\omega_\theta$ . The actual, corrected frequencies will be denoted by an asterisk in order to distinguish them from the corresponding eigenfrequencies:  $\nu_r^* \equiv (\omega_r + \Delta\omega_r)/(2\pi)$ ,  $\nu_\theta^* \equiv (\omega_\theta + \Delta\omega_\theta)/(2\pi)$ . Mathematical validity of the adopted approximation requires the frequency corrections to be small in our model.

Equations (8) and (9) can be solved for  $\sin \gamma$  and integrated to read

$$a_r^2 + \frac{\alpha \omega_r}{\beta \omega_\theta} a_\theta^2 \equiv E = \text{const.} \quad (12)$$

The quantity  $E$  is proportional to the energy of oscillations and is conserved in this order of approximation. Equation (12) describes an ellipse in  $(a_r, a_\theta)$  coordinates and it rules the flow of energy between the oscillation modes.

The two epicyclic frequencies  $\omega_r$  and  $\omega_\theta$  define the basic modes of oscillations. Secular terms and the possibility of parametric resonance appear in the second order of approximation. In the third order, the frequency corrections are characterized by the foursome of parameters:  $\kappa_r$ ,  $\kappa_\theta$ ,  $\lambda_r$ , and  $\lambda_\theta$ . Finally, the two additional parameters,  $\alpha$  and  $\beta$ , provide the fourth-order corrections, and describe the semi-axes ratio of the energy ellipse. The above-given list completes the most general description of any system described by equations of the forms (1) and (2).

The general solution corresponds to the periodic exchange of energy between the two oscillators. If the system is far from the steady state, the amplitudes and frequencies of oscillations fluctuate, maintaining the energy condition (12). The period of energy exchange is approximately given by

$$T \sim \frac{16\pi}{\beta \omega_\theta} E^{-3/2}. \quad (13)$$

On the other hand, particular steady-state solutions are possible in which the observed frequencies of the two modes remain strictly constant in time in 3 : 2 ratio, despite the fact that the eigenfrequencies depart from it. Such solutions play a role of singular points of the system (8)–(11). Close to steady state the approximation (13) fails and the period becomes greater (Horák 2004).

In order to verify accuracy of our work, we checked that the numerical solution of a special form (Abramowicz et al. 2003) of the system of equations (1) and (2), in which the right-hand side corresponded to the case of nearly geodesic circular motion in the pseudo-Newtonian potential of Paczyński and Wiita (1980), does indeed closely follow the ellipse of equation (12). Therefore we can be confident that the above-described analytical method gives credible results.

It is evident from equations (8), (9), (12), and (13), that the amplitudes and frequencies of high-frequency QPOs are modulated at the frequency  $\nu_3 \equiv 1/T$ , which is related to the amplitudes of the original high-frequency QPOs. We remind the reader that here we restrict ourselves to the case  $\omega_\theta : \omega_r = 3 : 2$ , however, similar results can be obtained also for other eigenfrequency ratios.

### 3. Low-Frequency Modulation

In the general discussion here, our model has six free parameters, corresponding to (a combination of) the lowest order expansion coefficients in the Taylor series of the unspecified functions  $f$  and  $g$  in equations (1) and (2). Hence, we cannot predict unique behavior. However, it is remarkable that a modulation of frequencies and amplitudes follows naturally from the assumptions made. Because the oscillators are nonlinear, their frequency varies with amplitude. Because the two oscillators are coupled in a system with constant energy, the amplitudes of the oscillators are anticorrelated. Because the two oscillators are in resonance, their frequencies are correlated.

It is not lost on us that similar correlations — discussed by Yu et al. (2001) — hold also on long timescales. For example, van der Klis et al. (1997) show that as the kHz frequencies increase, the ratio of power in the upper to lower kHz QPOs

decreases (up to a point). Our figure 1 demonstrates this to be the case also here. We note that in previous work, a particular relation was found to hold between the two frequencies of the system, in agreement with that observed in long-term variations of the twin kHz QPO frequencies in Sco X-1 (Abramowicz et al. 2003; Rebusco 2004). Here, we consider variations on shorter timescales.

As we mentioned above [see equation (13)], the timescale of modulation in our model is directly related to energy  $E$ , i.e., the weighted sum of amplitudes squared, which we adjusted to obtain a modulation at the NBO frequency of Sco X-1,  $\nu_3 \approx 6$  Hz. We identify the corrected frequencies of our model, i.e.,  $\nu_r^*$  and  $\nu_\theta^*$ , with the twin kHz QPO frequencies  $\nu_{\text{lower}}$  and  $\nu_{\text{upper}}$ , respectively.

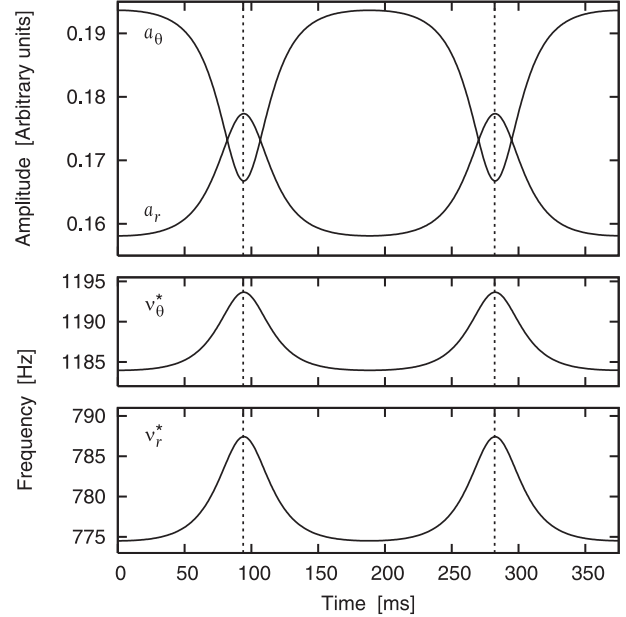
Figure 1 (top panel) exhibits the time variation of the “radial” and “vertical” amplitudes of oscillations. In the lower panels we show the correlation between the two kHz frequencies found in our solution. By the assumption of nearly 3 : 2 resonance, the two frequencies of actual oscillations satisfy relation  $\nu_{\text{upper}}^* \approx 1.5\nu_{\text{lower}}^*$ . Notice a perfect correlation between the variation of the lower amplitude and the variation of the upper frequency  $\nu_{\text{upper}}^*$ . This we interpret as the same correlation that was found by Yu et al. (2001); compare their figure 2. The magnitude of frequency variation agrees to within a factor of 3 with the data: it is 20 Hz for  $\nu_{\text{upper}} = 1.1$  kHz in the data, and about 7 parts in 1000 in our calculation.

#### 4. Discussion

We have found that a non-linear resonance in a system simulating an accretion disk, invoked previously to explain the appearance of two frequencies in approximately 3 : 2 ratio in black-hole and neutron-star X-ray data, results in a periodic time variation of the frequencies.

The Fourier transform of the frequency-modulated and amplitude-modulated periodic signal would result in several sidebands, in practice leading to an increase in width of the (noisy) signal. If the observed radiation flux were modulated with the squared modulus of the amplitudes (5) and (6), its Fourier transform would exhibit a (weak) component at low frequency of the modulation apparent in figure 1, in addition to the high-frequency signal with its sidebands. In this exploratory work, we are not attempting to model the full power spectrum of QPO sources and, as yet, we have made no attempt to translate the amplitudes of motion in the model into modulations of the X-ray flux. Strictly speaking, our toy-model gives a coherent signal rather than a QPO, and no details of the excitation or damping were modelled.

It has been suggested that the high-frequency QPOs vary on a timescale of seconds in some sources, notably in Sco X-1 (Yu et al. 2001) and in the black hole candidate XTE J1550–564 (Yu et al. 2002). The low-frequency modulation occurs at  $\approx 6$  Hz in both sources. In the above described calculation we were able to reproduce this variation for Sco X-1, including the anti-correlation between the amplitude of the lower peak and the frequency of the upper one. A similar approach is possible also in case of XTE J1550–564, although the black hole candidates typically exhibit lower frequencies compared to those in



**Fig. 1.** Time evolution of amplitudes (top panel) and the epicyclic frequencies (vertical in the middle, radial at bottom) of high-frequency QPOs. The two epicyclic modes are interpreted as radial/vertical oscillations of the accreting fluid. The variation of the oscillation amplitudes is not arbitrary in the adopted resonance model: instead, a relation between the amplitudes and frequencies is naturally predicted.

neutron stars, and particularly to those in Sco X-1. This means that if the 6 Hz QPO seen by Yu et al. (2002) corresponds to the modulation discussed in our model, then the ratio  $\nu_{\text{lower}}/\nu_3$  has to be set differently (about 30 in XTE J1550–564 with  $\nu_{\text{lower}} = 184$  Hz). And this in turn implies  $E$  is different in both systems.

If the correspondence of our results with the observed modulation of kHz QPO properties on the 6 Hz NBO timescale is not accidental, for the first time we would have a physical explanation for the presence of the rather low frequency QPO in what is otherwise a domain of rapid variability. The larger point is that non-linear resonance (most likely between modes of oscillation possible only in strong gravity) holds promise for explaining not only the highest frequencies observed in accreting neutron stars and black holes, but also the mysterious phenomenology of low frequency features in the power density spectrum, without invoking additional mechanisms.

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