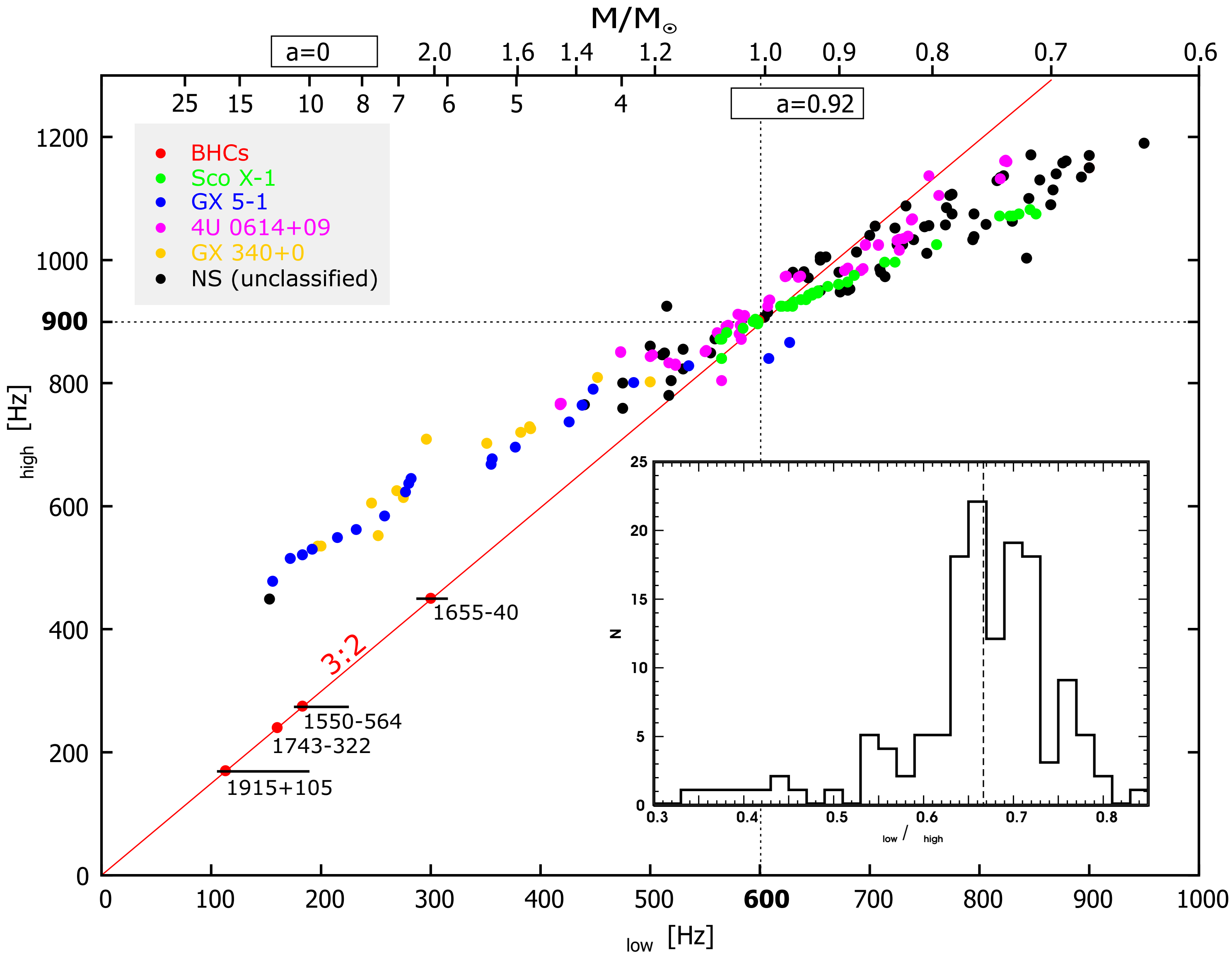
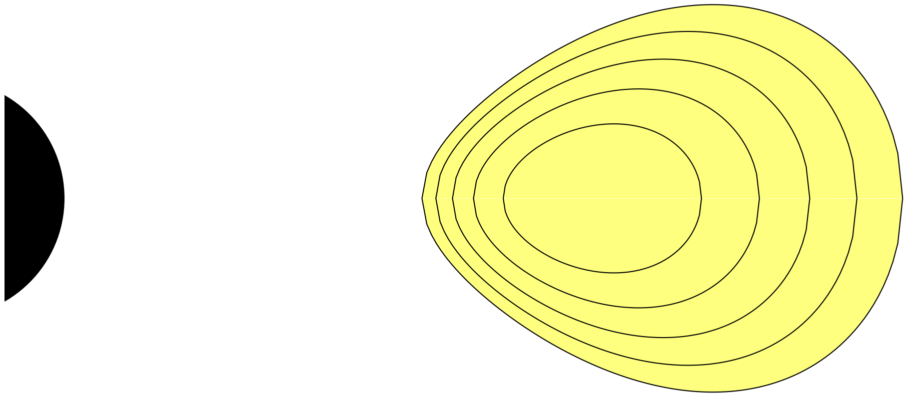
Signatures of **nonlinear** *resonances*

QPO: Simple physics?



What oscillates?



$$\begin{aligned}\delta\ddot{r} + \omega_r^2 \delta r &= \alpha_r(\delta r, \delta\dot{r}) + f_r(\delta r, \delta\theta, \delta\dot{r}, \delta\dot{\theta}) + F_r(t),\\ \delta\ddot{\theta} + \omega_\theta^2 \delta\theta &= \alpha_\theta(\delta\theta, \delta\dot{\theta}) + f_\theta(\delta r, \delta\theta, \delta\dot{r}, \delta\dot{\theta}) + F_\theta(t).\end{aligned}$$

General conditions:

- Symmetry with respect to the equatorial plane: $f_r(\delta\theta) = f_r(-\delta\theta), f_\theta(\delta\theta) = -f_\theta(-\delta\theta)$
- f_r and f_θ remain unchanged when $\mathrm{d}/\mathrm{d}t \to -\mathrm{d}/\mathrm{d}t$.

Nonlinear oscillations

Quadratic nonlinearity:

$$\ddot{x} + \omega^2 x = \alpha \omega^2 x^2$$

A simple perturbation method:

$$x(\epsilon; t) = \epsilon x_1(t) + \epsilon^2 x_2(t) + \ldots$$

leads to the system

$$\ddot{x}_n + \omega^2 x_n = f(x_{n-1}, x_{n-2}, \ldots, x_1)$$

that produces non-uniform expansions.

Nonlinear oscillations

A more sophistikated method are the **multiple scales**

$$t \longrightarrow T_\mu = \epsilon^\mu t \quad \mu = 0, 1, 2, \ldots$$

we assume

$$x(\epsilon; t) \longrightarrow x(\epsilon T_\mu) = \epsilon x_1(T_\mu) + \epsilon^2 x_2(T_\mu) + \ldots$$

This leads to

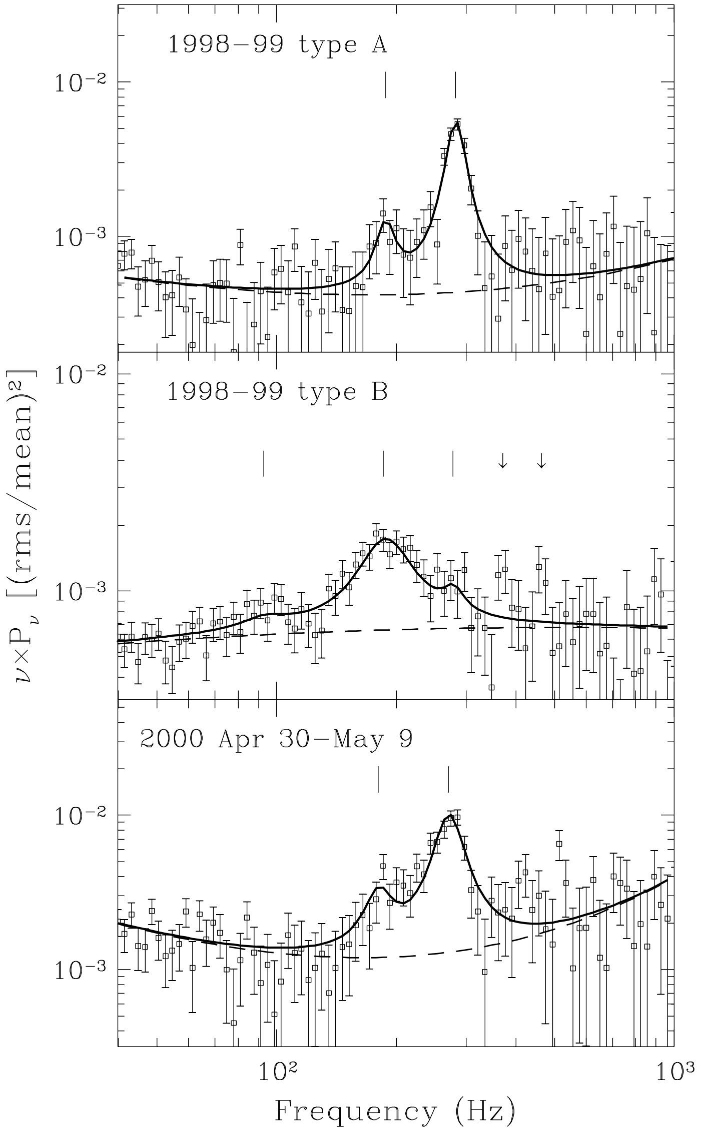
$$\left[\frac{\partial^2}{\partial T_0^2} + \omega^2\right] x_n = f(x_{n-1}, x_{n-2}, \ldots)$$

Hence

$$x_1 = A(T_1, T_2, \ldots) e^{\mathrm{i}\omega T_0} + \mathrm{cc}$$

A slow evolution of the phases and amplitudes.

Nonlinear oscillations



- Solution up to the first order

$$x_1 = A(t)e^{\mathrm{i}\omega T_0} + \mathrm{cc}$$

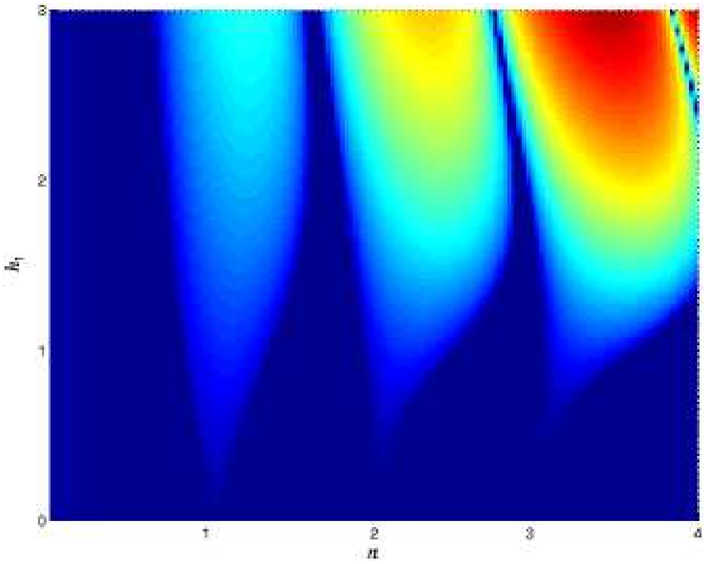
where $A(t) = a(t)e^{\mathrm{i}\phi(t)}$ is a slowly varying complex amplitude

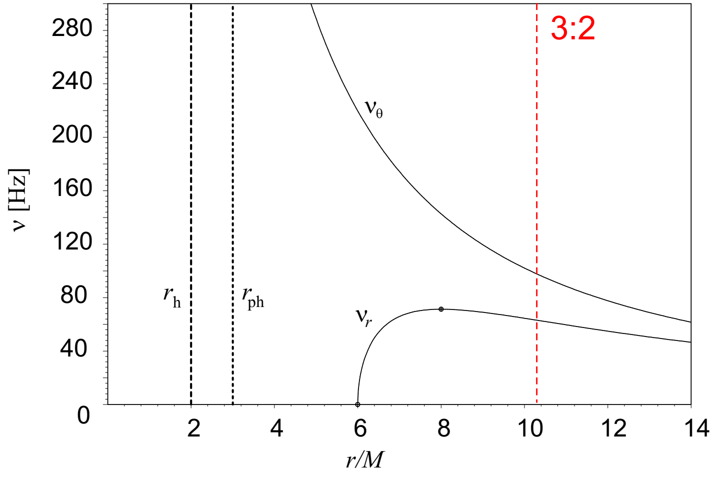
- True (observed) frequency of oscillations is given as

$$\omega^\star = \omega + \dot{\phi}$$

- Higher order solutions x_2, x_3, ... give us the **harmonics**

Coupling: Autoparametric resonance





- Small radial oscillations:
  $\delta r = A_r \cos(\omega_r t)$,

  $$\delta\ddot{\theta} + \omega_\theta^2 \left[1 + \alpha A_r \cos(\omega_r t)\right] \delta\theta = 0,$$

  Parametric resonances: $\frac{\omega_\theta}{\omega_r} = \frac{n}{2}$,
- Small vertical oscillations:
  $\delta\theta = A_\theta \cos(\omega_\theta t)$,

  $$\delta\ddot{r} + \omega_r^2 \delta r = \alpha \delta r^2 \delta\theta^2 \propto \cos(2\omega_\theta - 2\omega_r) + \ldots$$

  The radial oscillations are forced by nonlinearity $\propto \delta r^2 \delta_\theta^2$.

Nonlinear dynamics

- Governing equations

$$\begin{aligned}\delta\ddot{r} + \omega_r^2 \delta r &= f_r(\delta r, \delta\theta, \delta\dot{r}, \delta\dot{\theta}),\\ \delta\ddot{\theta} + \omega_\theta^2 \delta\theta &= f_\theta(\delta r, \delta\theta, \delta\dot{r}, \delta\dot{\theta}).\end{aligned}$$

- Solutions are of the form

$$\begin{aligned}\delta r &= A_r(t)e^{\mathrm{i}\omega_r t} + \mathrm{cc},\\ \delta\theta &= A_\theta(t)e^{\mathrm{i}\omega_\theta t} + \mathrm{cc}\end{aligned}$$

  where $A_t \equiv a(t)e^{i\phi}$ is *slowly* variing complex amplitude.

- Frequency corrections: $\Delta\omega \equiv \omega^\star - \omega = \dot{\phi}$

Nonlinear dynamics

Equations describing behaviour of amplitudes and phases

$$
\begin{aligned}
\dot{a}_r &= \frac{\alpha\omega_r}{16}\, a_r^2\, a_\theta^2\, \sin\gamma, \\
\dot{a}_\theta &= -\frac{\beta\omega_\theta}{16}\, a_r^3\, a_\theta\, \sin\gamma, \\
\dot{\phi}_r &= -\frac{\omega_r}{2}\, \left[\kappa_r\, a_r^2 + \kappa_\theta\, a_\theta^2\right] - \frac{\alpha\omega_r}{16}\, a_r\, a_\theta^2\, \cos\gamma, \\
\dot{\phi}_\theta &= -\frac{\omega_\theta}{2}\, \left[\lambda_r\, a_r^2 + \lambda_\theta\, a_\theta^2\right] - \frac{\beta\omega_\theta}{16} a_r^3\, \cos\gamma,
\end{aligned}
$$

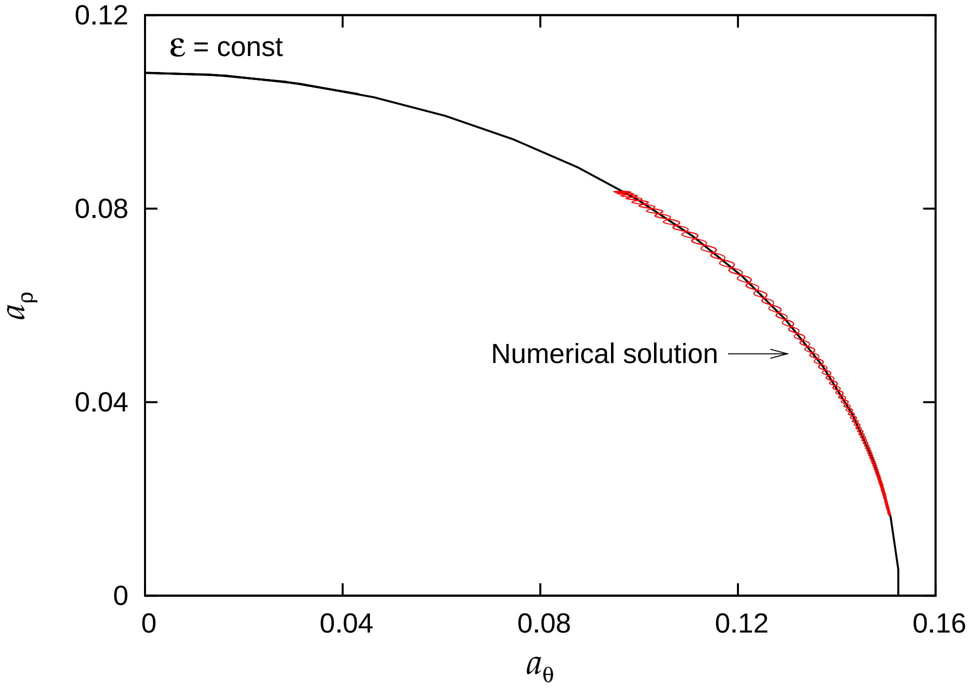
- $\gamma \equiv 2(\phi_\theta + \omega_\theta t) - 3(\phi_r + 2\omega_r t)$ is phase functions
- Parameters α, β, κ_r, λ_r, κ_θ, λ_θ depend on the properties of the system (Taylor expansion of functions f_r and f_θ).

Integral of motion

The first two equations for $\dot{a}_\rho(t)$ and $\dot{a}_\theta(t)$ imply

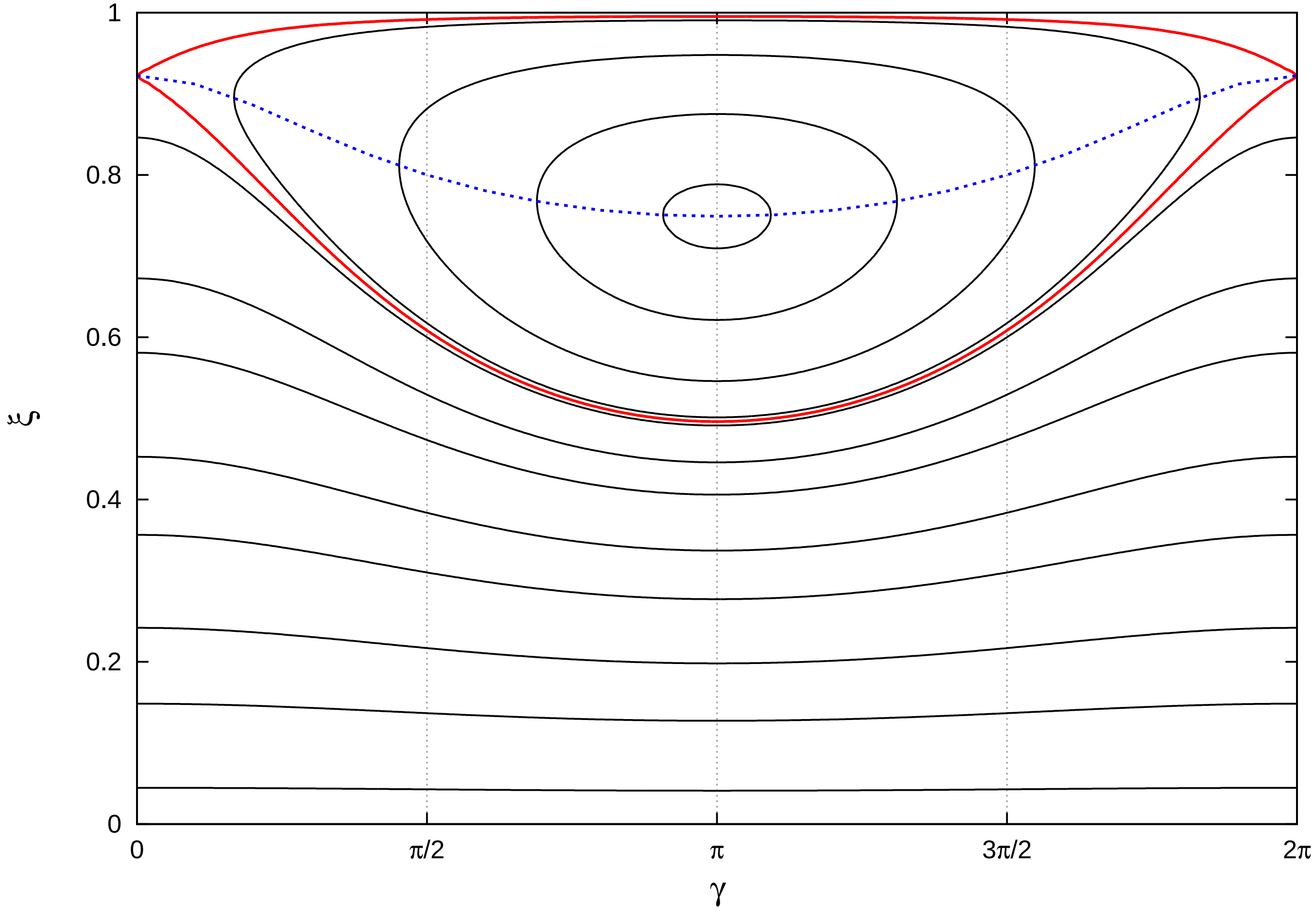
$$\frac{d}{dt}(a_r^2 + \nu a_\theta^2) = \frac{d\mathcal{E}}{dt} = 0 \quad \nu \equiv \frac{2\alpha}{3\beta}$$

- The system is conservative up to the 4th order.



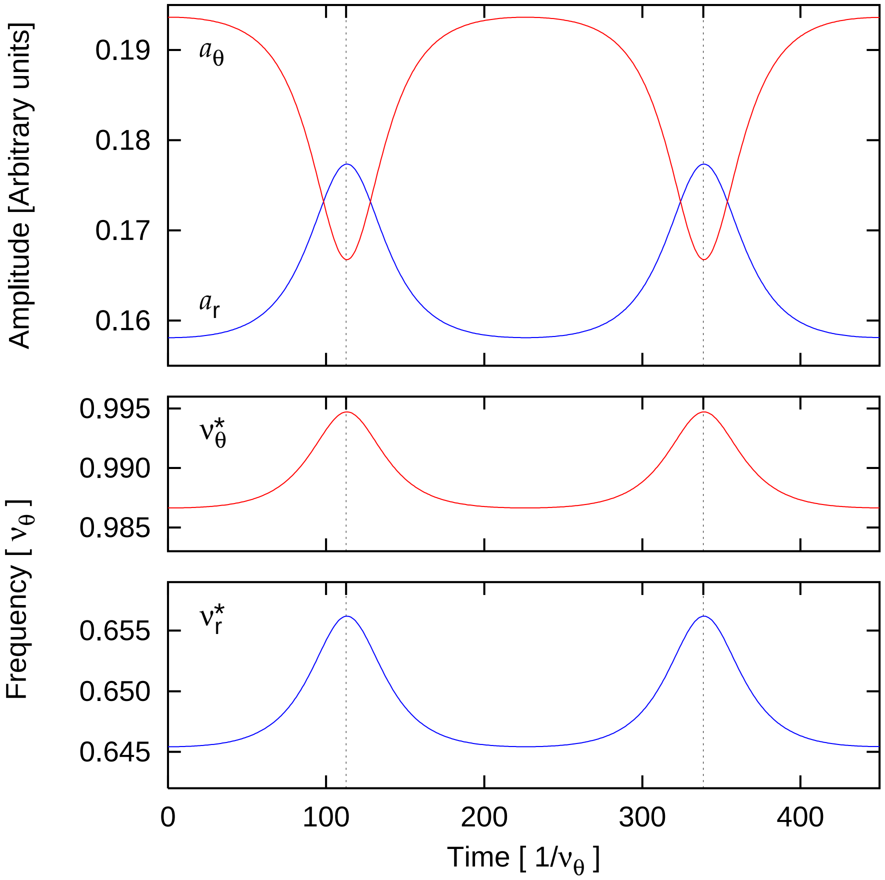
- $a_r = \xi \mathcal{E}^{1/2}$,
- $a_\theta = \sqrt{1-\xi^2}\left(\frac{\mathcal{E}}{\nu}\right)^{1/2}$

Frequencies of resonant oscillations



$$\dot{\gamma} = \frac{d}{dt}\left[2(\phi_\theta + \omega_\theta t) + 3(\phi_r + \omega_r t)\right] = 2\omega_\theta^\star - 3\omega_r^\star$$

Amplitudes and frequencies of resonant oscillations

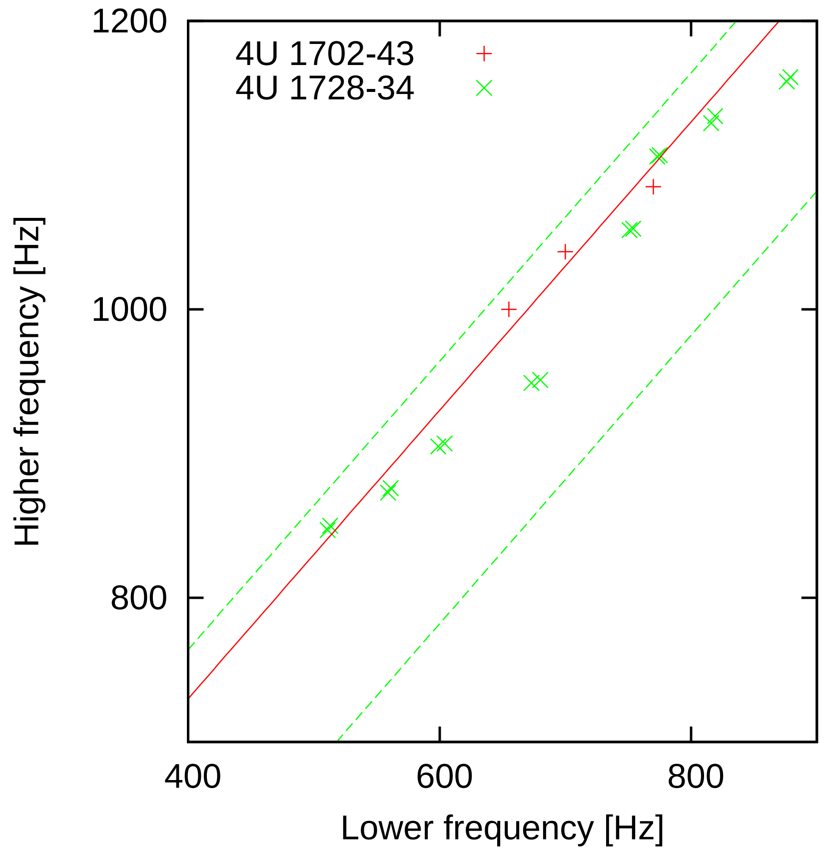


- Anti-correlation of amplitudes
- Correlation of frequencies
- Period of the modulation

$$T \sim \frac{16\pi}{\beta\omega_\theta} E^{-3/2}$$

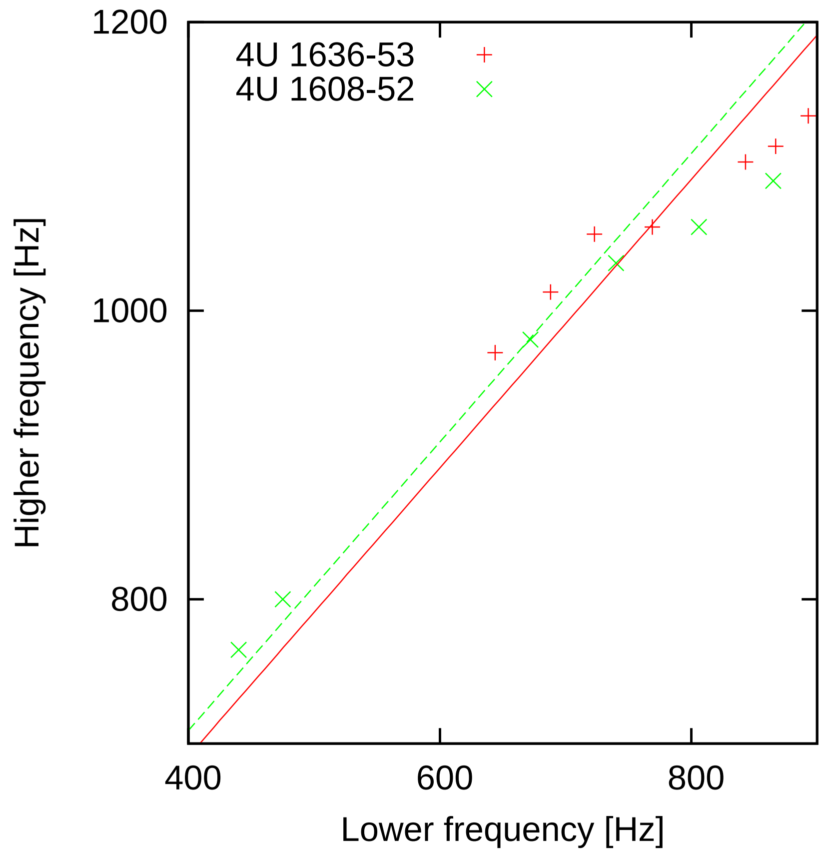
Forced oscillations in an orbital motion

Slow rotators:



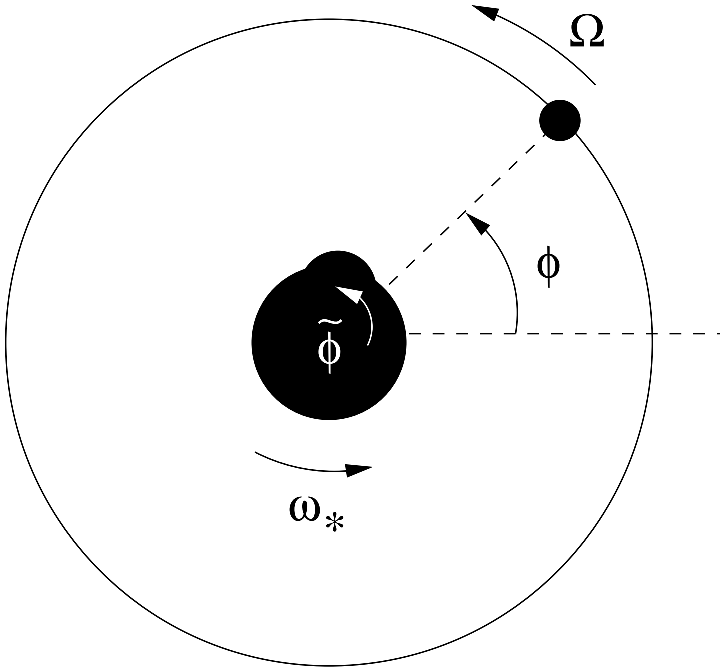
$$\Delta\nu = \nu_2 - \nu_1 \approx \nu_\star$$

Fast rotators:



$$\Delta\nu = \nu_2 - \nu_1 \approx \nu_\star/2$$

Very simple model



The force on the particle:

$$F^j \equiv F^j(\tilde{\phi}) = \sum_{m>0} k^j_m e^{\mathrm{i}m\tilde{\phi}},$$

where

$$\tilde{\phi} = \phi - \omega_\star t = (\Omega - \omega_\star)t$$

Then

$$F^j = \sum_m k^j_m e^{\mathrm{i}m(\Omega - \omega_\star)t}.$$

$$\ddot{r} - r\dot{\theta}^2 + \frac{\partial \mathcal{U}}{\partial r} = F^r(t), \quad \ddot{\theta} + 2\,\frac{\dot{r}\dot{\theta}}{r} + \frac{1}{r^2}\frac{\partial \mathcal{U}}{\partial \theta} = F^\theta(t).$$

Forced radial oscillations

We assume

$$\delta\theta = 0, \qquad F^{\theta} = 0.$$

Governing equation

$$\delta\ddot{\rho} - \mu\delta\dot{\rho} + \omega_r^2\delta\rho + \alpha_2\delta\rho^2 + \alpha_3\delta\rho^3 = \frac{1}{r_0}\sum_m k_m^r e^{i(\Omega-\omega_\star)t}$$

where

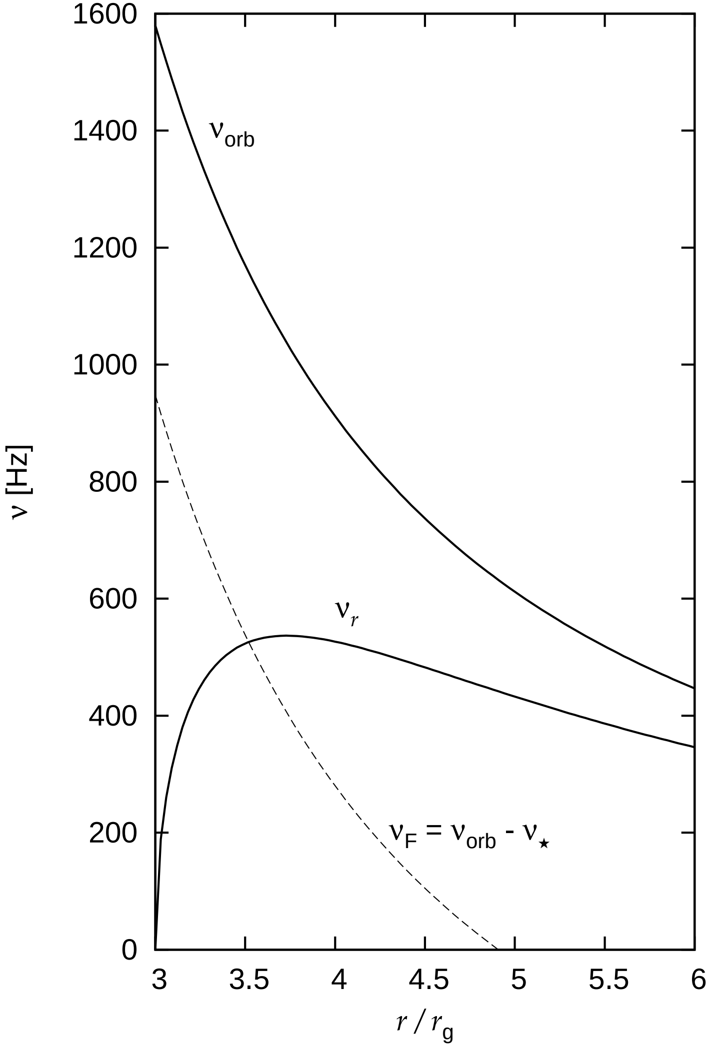
$$\delta\rho = \frac{r - r_0}{r_0}$$

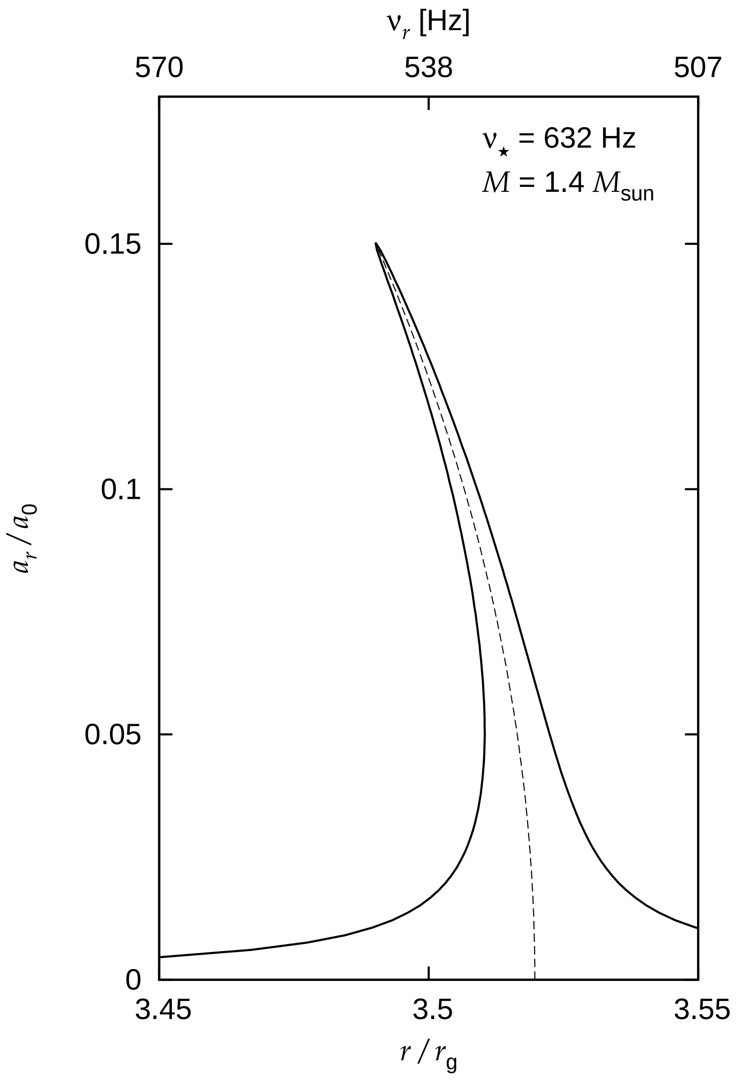
$$\alpha_2 = \frac{1}{2r_0}\left(\frac{\partial^3\mathcal{U}}{\partial r^3}\right)_0, \qquad \alpha_3 = \frac{1}{6r_0^2}\left(\frac{\partial^4\mathcal{U}}{\partial r^4}\right)_0.$$

Resonance condition

$$m = 1: \qquad \Omega - \omega_\star = \omega_r$$

Forced radial oscillations





Forced vertical oscillations

We assume

$$\delta\rho = 0 \quad F^r = 0.$$

Governing equation

$$\delta\ddot{\theta} + \omega_\theta^2 + \alpha\delta\theta^3 = \frac{1}{r_0}\sum_m k_m^\theta e^{im(\Omega-\omega_\star)t},$$

where

$$\alpha = \frac{1}{6r^2}\left(\frac{\partial^4\mathcal{U}}{\partial\theta^4}\right)_0$$

Resonance condition

$$m(\Omega-\omega_\star) = \omega_r, \quad m = 2, 3, \ldots$$

Forced vertical oscillations