Introduction

Stationary purely azimuthal flow

$$\frac{\ell^2}{r^3}\vec{r} = \frac{\nabla p}{\rho} + \nabla\Phi$$

- Points where $\nabla p = 0$ correspond to $\ell = \ell_{\rm Kepler}$
- $\ell(r) \equiv \ell_0 \rightarrow$ Effective potential: $\mathcal{U} \equiv \Phi + \frac{\ell_0^2}{2r^2}$
- $P = P(\rho) \rightarrow$ Enthalpy: $h \equiv \int_0^P \frac{dP}{\rho}$

Bernoulli equation:

$$h + \mathcal{U} = \text{const} \equiv \mathcal{U}_{\rm S}$$

- Surfaces of constant pressure and density coincide with the surfaces of constant effective potential.

General relativity

General relativistic dynamics can be modelled introducing the pseudo-Newtonian potential [a]

$$\Phi(R) = -\frac{GM}{R - R_{\rm g}}, \quad R_{\rm g} \equiv \frac{2GM}{c^2}$$

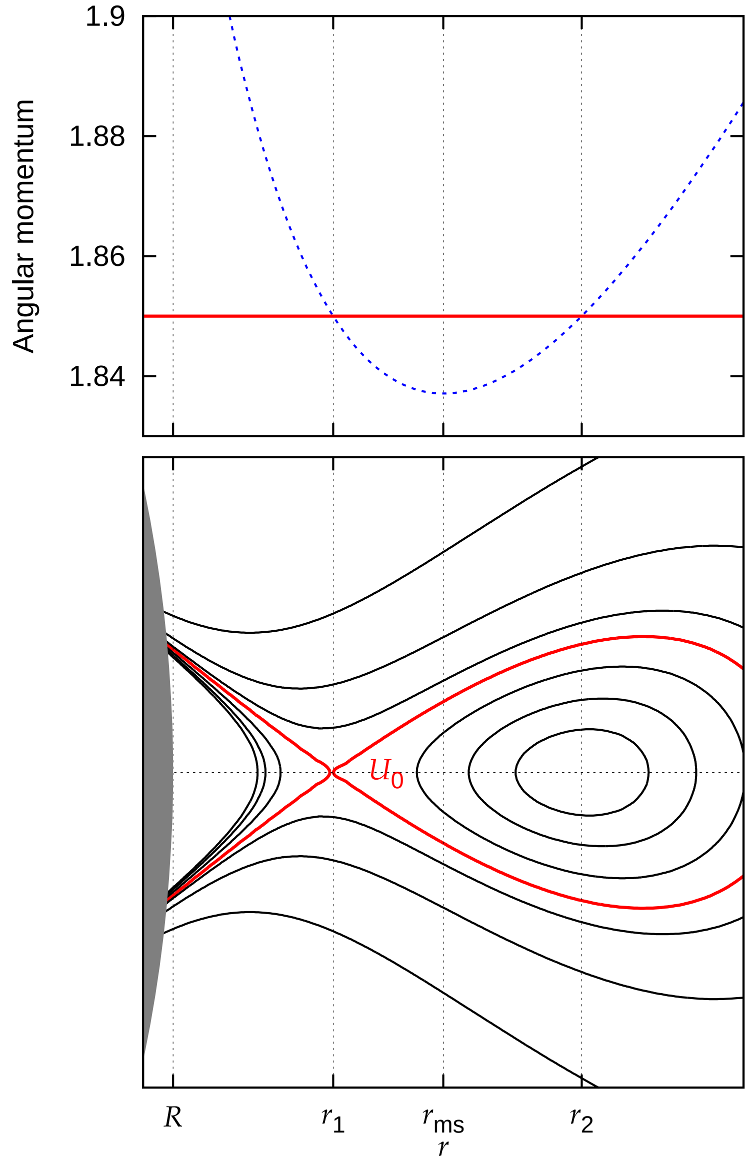
well reproduces the behaviour of the Keplerian angular momentum in general relativity

$$l_{\rm Kepler} = \frac{r}{r - R_{\rm g}}\sqrt{GMr}$$

- Minimum at marginally stable orbit $r_{\rm ms} = 3R_{\rm g}$

[a] Paczynski & Wiita (1980) A&A 88/23)

Paczynski's accretion scenario ($R_\star < r_{\rm ms}$)



1. Angular momentum ℓ_0 gives
   - the torus center r_1
   - the cusp r_0
   - Values of $\mathcal{U}_0$ and $\mathcal{U}_1$
2. Filling by the matter
3. The accretion rate depends on the Roche-lobe overflow

$$\Delta\mathcal{U} \equiv \mathcal{U}_{\rm S} - \mathcal{U}_0$$

- $\Delta\mathcal{U} < 0 \rightarrow$ no accretion

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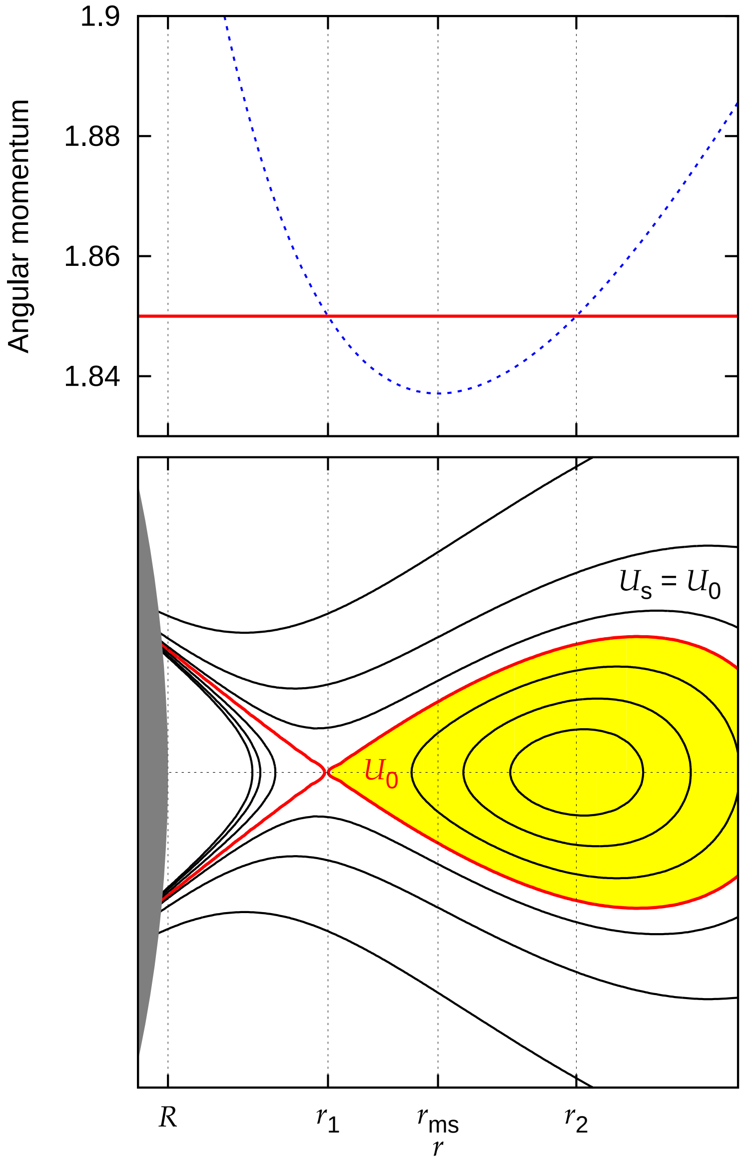

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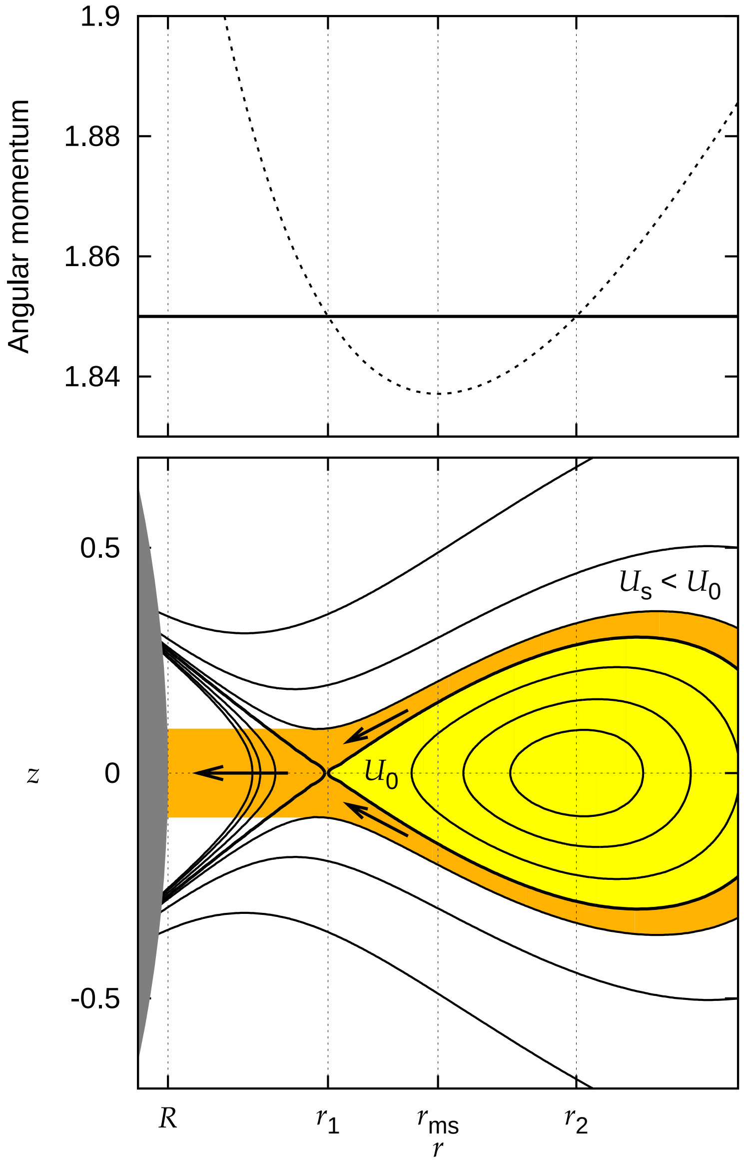

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Difference between BHs and NSs

Black holes:

- Event horizon at $r = R_{\rm g}$
- Flow properties below $r_{\rm ms}$ does not much affect the observed X-ray luminosity

Neutron stars:

- Stellar surface at $R_{\rm g} \leq R_\star \leq r_{\rm ms}$
- Under $r_{\rm ms}$ much of the accretion energy is released [a] (69% when $R_\star = 3R_{\rm g}$, 86% when $R_\star = 1.5R_{\rm g}$)
- *The stability of X-ray luminosity is governed by the stability of the accretion rate near* $r_{\rm ms}$ [b]

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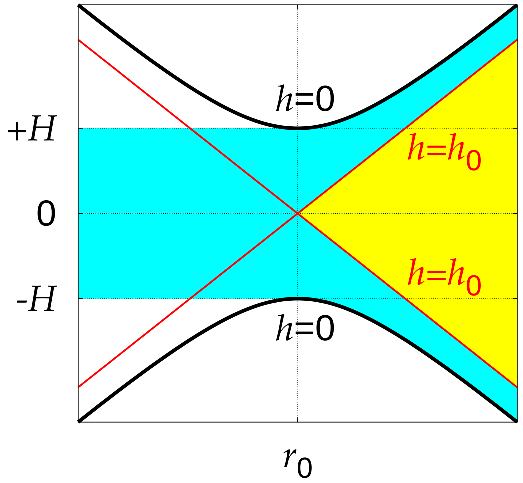
[a] Syunyaev & Shakura (1986) Sov.Astron.Lett 12/120

[b] Paczynski (1987) Nature 327/303

Accretion rate in stationary disk (I)

Our assumptions



- small overflow:

$$h \doteq h_0 - \frac{1}{2}\kappa^2 z^2, \qquad \kappa^2 \equiv -\frac{\partial^2 h}{\partial z^2}, \qquad H = \frac{\sqrt{2h_0}}{\kappa}$$

- barotropic fluid: $p = K\rho^{1+1/n}$
- accretion with the local sound speed

$$v^z \ll v^r \doteq c_\mathrm{s} = \sqrt{\frac{h}{n}}$$

The local mass flux

$$\dot{m} = \rho v^r = \rho c_\mathrm{s} = \frac{h^{n+1/2}}{K^n (1+n)^n n^{1/2}}$$

Accretion rate in stationary disk (II)

- The total mas flux in terms of the enthalpy

$$\begin{aligned}\dot{M} &= 4\pi r_0 \int_0^H \dot{m} dz = \\ &= (2\pi)^{3/2} \frac{r_0}{\kappa n^{1/2}} \left[\frac{1}{K(n+1)}\right]^n \frac{\Gamma(n+3/2)}{\Gamma(n+2)} h_0^{n+1}\end{aligned}$$

- Correspondence between h_0 and $\Delta\mathcal{U}$: Bernoulli equation

$$\frac{v^2}{2} + h + \mathcal{U} = \mathcal{U}_\mathrm{S}, \quad v^2 \doteq c_\mathrm{s}^2 = \frac{h}{n}$$

Therefore: $\Delta\mathcal{U} = \left(1 + \frac{1}{2n}\right) h, \quad \kappa^2 \equiv -\frac{\partial^2 h}{\partial z^2} = \left(\frac{n}{n+1/2}\right) \omega_z^2$

Accretion rate in stationary disk (III)

- Result: [a]

$$\dot{M} = A(n)\frac{r_0}{\omega_z}\Delta\mathcal{U}^{n+1}$$

where

$$A(n) \equiv (2\pi)^{3/2}\left[\frac{1}{K(n+1)}\right]^n\left[\frac{1}{n+1/2}\right]^{n+1/2}\frac{\Gamma(n+3/2)}{\Gamma(n+2)}$$

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[a] Abramowicz (1985) PASJ 37/727

Non-stationary perturbation

- Poloidal flow is potential $\vec{v} = \nabla\chi$
- Non-stationary Bernoulli equation

$$\frac{\partial\chi}{\partial t} + \frac{v^2}{2} + h + \mathcal{U} = \mathcal{U}_{\mathrm{S}}$$

- The velocity perturbations: $\chi(\boldsymbol{r}, t) = \chi_{(0)}(\boldsymbol{r}) + \epsilon\chi_{(1)}(\boldsymbol{r}, t)$

$$\begin{aligned} v^2 &= v_{(0)}^2 + 2\epsilon \boldsymbol{v}_{(0)} \cdot \boldsymbol{v}_{(1)} + \epsilon^2 v_{(1)}^2 \\ &= c_{\mathrm{s}}^2 + 2\epsilon c_{\mathrm{s}} \frac{\partial\chi_{(1)}}{\partial r} + \epsilon^2 \left[\left(\frac{\partial\chi_{(1)}}{\partial r} \right)^2 + \left(\frac{\partial\chi_{(1)}}{\partial z} \right)^2 \right]. \end{aligned}$$

- The enthalpy perturbation

$$h = h_{(0)} + \epsilon h_{(1)} + \epsilon^2 h_{(2)} + \mathcal{O}(\epsilon^3).$$

Non-stationary perturbation

- We get

$$
\begin{aligned}
h_{(1)} &= -\frac{\partial \chi_{(1)}}{\partial t} - \left(\frac{\bar{h}}{n}\right)^{1/2} \frac{\partial \chi_{(1)}}{\partial r}, \\
h_{(2)} &= -\frac{1}{2}\left[\left(\frac{\partial \chi_{(1)}}{\partial r}\right)^2 + \left(\frac{\partial \chi_{(1)}}{\partial z}\right)^2\right].
\end{aligned}
$$

What is $\delta\chi$ (I)?

Difficult to answer (global problem) $\rightarrow$ numerical simulations

Inspirations:

1. Slender tori of Omer Blaes

- Global modes of oscillations (analytic solution): [a]

$$\begin{aligned}\chi_r &= -iC_r(r - r_{\rm c})e^{i(m\phi - \Omega_r t)} \\ \chi_z &= -iC_z z e^{i(m\phi - \Omega_z t)}\end{aligned}$$

  Ω_r and Ω_z are epicyclic frequencies at the torus center $r_{\rm c}$

- $\mathcal{U}$ is expanded to the 2nd order $\rightarrow$ Tori are too slender

[a] Blaes (1985) MNRAS 216/553

What is $\delta\chi$? (II)

2. Larger tori of Luciano Rezzolla

- no accretion
- the radial axisymmetric p-mode oscillations [a]
- "...ω_r at the torus center represents the value at which the fundamental p-mode frequency tends in the limit of vanishing torus size."
- Checked by Eduardo Rubio-Herrera and William Lee. [b]

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[a] Rezzolla et al (2003) MNRAS 344/L37

[b] Rubio-Herrera & Lee (2004) astro-ph/0411654

What is $\delta\chi$? (III)

3. Geodesic deviation argument of Marek Abramowicz and Wlodek Kluzniak

- Synge [a]: A world-tube of an isolated body contains a geodesic line.
- Thomas [b], Taub [c]: When the body is formed by a barotropic fluid, this geodesics is a world-line of the maximal pressure point.
- Possible small oscillations of the body contains modes governed by the geodesic-deviation equation.
- Epicyclic oscillations are a general property of fluid bodies [d].

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[a] Synge (1960) “Relativity: the general theory”

[b] Thomas (1962) Proc N.A.S 1567

[c] Taub (1962) Proc N.A.S, 1570

[d] Blaes et al (2005) in preparation

Modulation by vertical epicyclic oscillations I.

Perturbation: $\chi_{(1)} = z v_z \cos \omega t$,

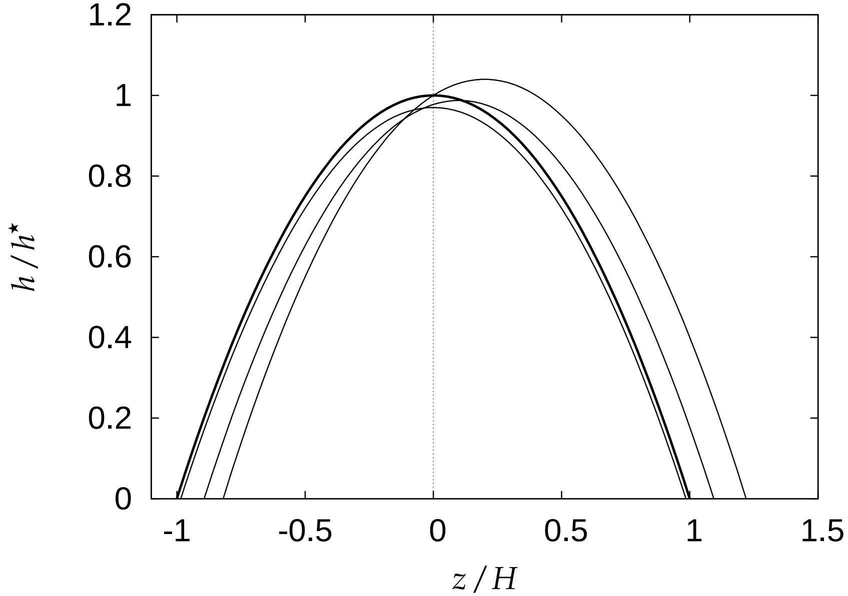
- Velocity perturbation: $\boldsymbol{v}_{(1)} = v_z \boldsymbol{e}_z \cos \omega t$
- Enthalpy perturbation

$$h_{(1)} = z v_z \omega \sin \omega t, \quad h_{(2)} = -\frac{1}{2} v_z^2 \cos^2 \omega t.$$

- Enthalpy profile - quadratic

$$\begin{aligned} h(r_{\mathrm{in}}, z, t) &= h^{\star} - \kappa^2 z^2 + \epsilon z v_z \omega \sin \omega t - \\ &- \tfrac{1}{2} \epsilon^2 v_z^2 \cos^2 \omega t + \mathcal{O}(\epsilon^3) \end{aligned}$$

Modulation by vertical epicyclic oscillations II.



- Maximal pressure point:

$$\begin{aligned} \delta z(t) &= \delta Z \sin \omega t, \\ \delta Z &= \epsilon \frac{\omega v_z}{\kappa^2}. \end{aligned}$$

- Maximal enthalpy:

$$\begin{aligned} \delta h_0 &\equiv h(r_{\rm in}, \delta z) - h_0 \\ &= \frac{1}{2}\kappa^2 \left[\delta z^2 - \frac{\kappa^2}{\omega^2}(\delta Z^2 - \delta z^2) \right] + \mathcal{O}(\epsilon^3). \end{aligned}$$

Modulation by vertical epicyclic oscillations III.

- Accretion rate

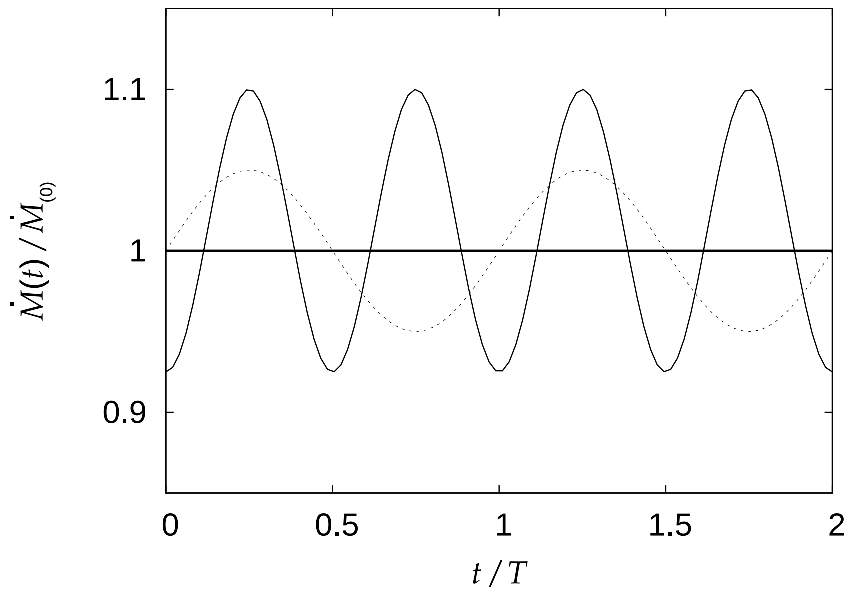
$$\dot{M} = (2\pi)^{3/2} \frac{r_0}{\kappa n^{1/2}} \left[\frac{1}{K(n+1)} \right]^n \frac{\Gamma(n+3/2)}{\Gamma(n+2)} h_0^{n+1}$$

Modulation of the accretion rate:

$$\frac{\delta \dot{M}}{\dot{M}_{(0)}} = (n+1) \frac{\delta h_0}{h_0} = \frac{2-p}{2-2p} \left[(1+p) \frac{\delta z^2}{H^2} - p \frac{\delta Z^2}{H^2} \right],$$

Resume

- Simple way to estimate modulation of accretion rate (X-ray luminosity) with the knowledge of dynamic of the accretion disk ($\delta\chi$)



- Modulation of accretion rate by vertical epicyclic oscillations:

$$\frac{\delta\dot{M}}{\dot{M}} \propto \left(\frac{\delta z_0}{H}\right)^2$$

- Averaged accretion rate:

$$\frac{\langle\delta\dot{M}\rangle}{\dot{M}_{(0)}} = \frac{1}{4}(2-p)\frac{\delta Z^2}{H^2}$$

Modulation by radial epicyclic oscillations I.

perturbation: $\delta\chi = r\delta v_r \cos(\omega_r t)$

- Velocity perturbation: $\delta\vec{v} = \delta v_r \cos(\omega_r t)\vec{e}_r$
- Enthalpy

$$h(t) = \bar{h}_0 - \frac{1}{2}\kappa^2 z^2 - \delta v_r \left(\frac{\bar{h}}{n}\right)^{1/2} \cos(\omega_r t)$$

Modulation by radial epicyclic oscillations II.