Nonlinear oscillations of fluid tori: a framework for explaining kHz QPOs?

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Outline

- Introduction – motivation

  $\longrightarrow$ Observing QPOs – nonlinear resonance?
- Nonlinear oscillations of fluid bodies

  $\longrightarrow$ Fourth-order Lagrangian perturbation theory

  $\longrightarrow$ Modal equation as many harmonic oscillators

  $\longrightarrow$ Effects of symmetry, selection rules

  $\longrightarrow$ Free oscillations, internal resonances
- Free oscillations of slender tori

  $\longrightarrow$ Parametric excitation

  $\longrightarrow$ Excitation by external forcing
- conclusions, open quastions

Introduction

Introduction: observing QPOs

Sources:

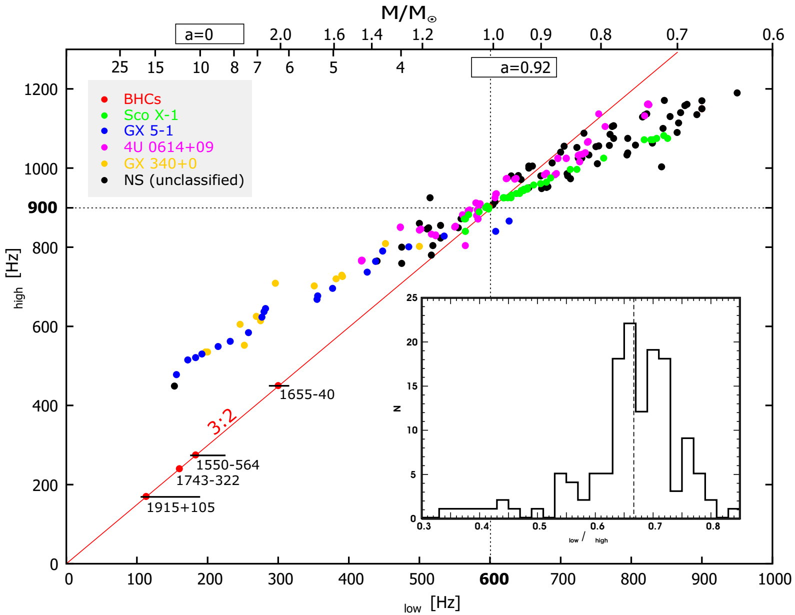
- High-frequency QPOs are observed in X-rays of Low Mass X-ray binaries
- observed in both black-hole and neutron star binaries
- Frequencies are comparable with Keplerian frequencies close to the compact object

$$\nu_{\rm ISCO} = 1580 \frac{M_\odot}{M} \,\mathrm{Hz}$$

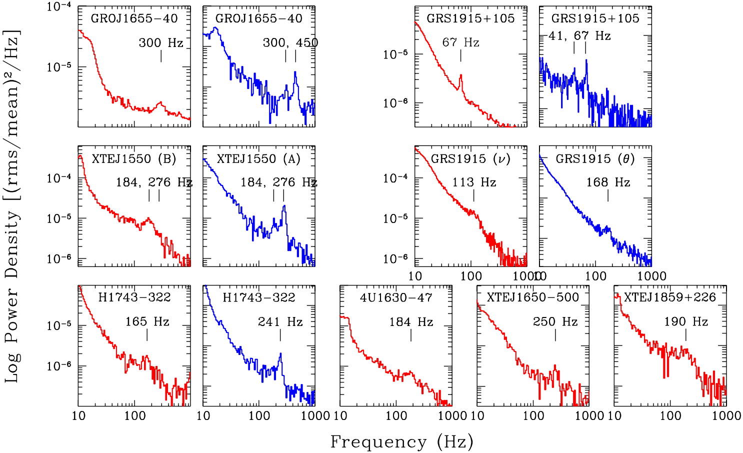
Frequency pairs (ν_ℓ and ν_u)

- Frequencies scales inversely with the mass of the compact object
- Frequencies are *stable* in black hole sources
- Rational (3:2) ratio

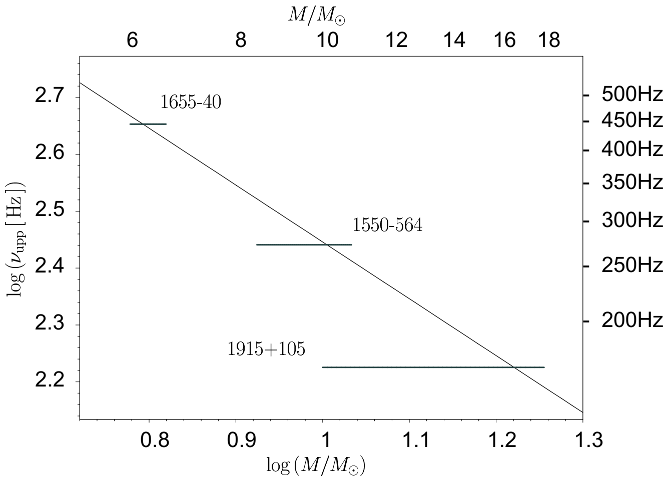
High-frequency QPOs (Bursa 2004)



Black-holes QPOs (McClintock & Remillard 2005)



Inverse mass scaling (McClintock & Remillard)



Resonance model (Abramowicz and Kluzniak)

1. What oscillates? – frequency identification:
    - Upper QPO frequency == Vertical epicyclic frequency
    - Lower QPO frequency == Radial epicyclic frequency
2. Nonlinear interaction of epicyclic oscillations → resonance:

$$\delta\ddot{r} + \omega_r^2 \delta r = f_r(\delta r, \delta\theta) \quad \delta\ddot{\theta} + \omega_\theta^2 \delta\theta = f_\theta(\delta r, \delta\theta)$$

Results: Expanding up to the second order in vertical equations,

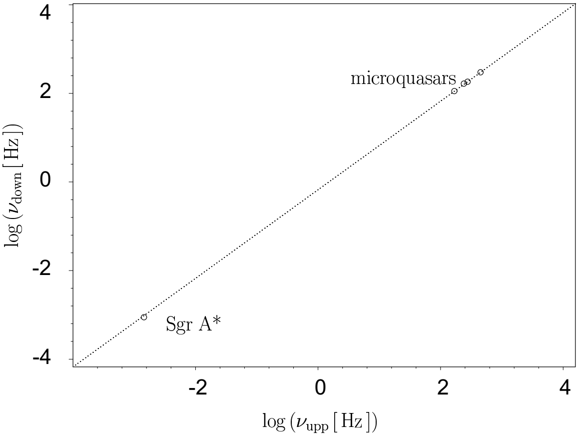
$$\delta\ddot{\theta} + \omega_\theta^2 (1 + k\delta r)\delta\theta = 0$$

and assuming small radial oscillations, $\delta r \propto \cos(\omega_r t)$, we get Mathieu equation. Parametric resonance (exponential grow of $\delta\theta(t)$) when

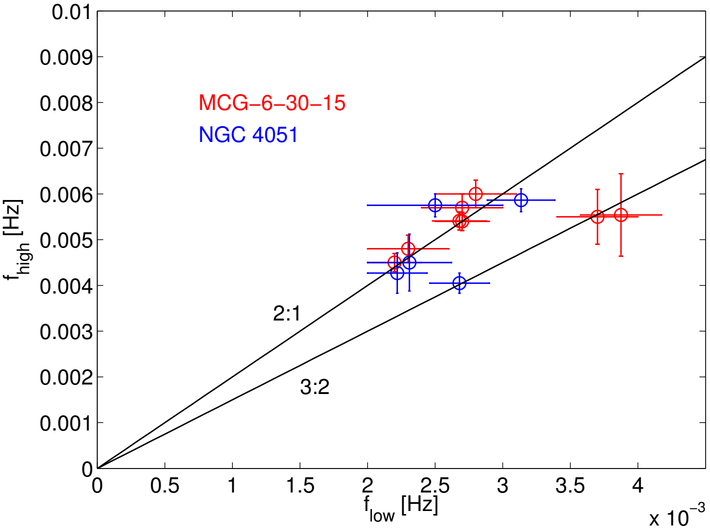
$$\frac{\omega_\theta}{\omega_r} = \frac{n}{2}$$

GR→ $\omega_\theta \geq \omega_r \rightarrow n = 3$ is the strongest resonance

QPOs from AGNs? (Torok 2005)



QPOs from AGNs? (Lachowicz et al 2006)



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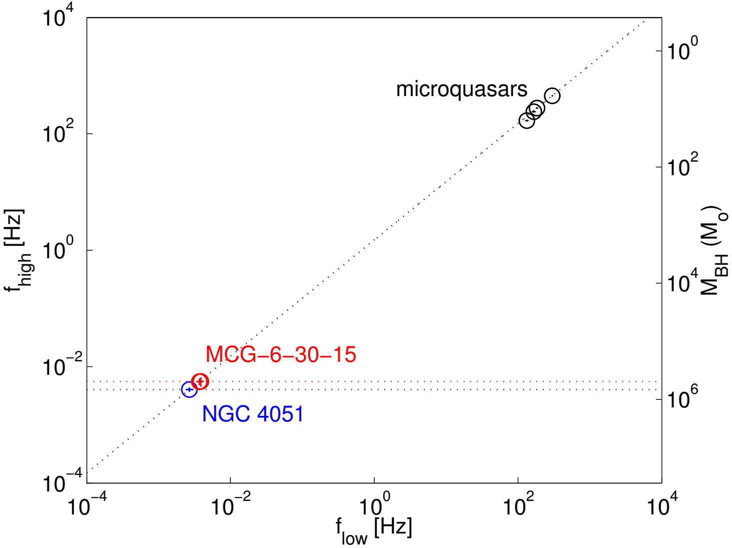


Nonlinear oscillations of fluid bodies

Lagrangian approach to perturbations

Eulerian and Lagrangian perturbations, Lagrangian displacement

$$\begin{aligned}\delta Q(\mathbf{x},t) &\equiv Q(\mathbf{x},t) - Q_0(\mathbf{x},t),\\ \Delta Q(\mathbf{x},t) &\equiv Q(\mathbf{x}+\boldsymbol{\xi},t) - Q_0(\mathbf{x},t)\end{aligned}$$

Lagrangian perturbations:

$$\Delta\mathbf{v} = \frac{\mathrm{D}\boldsymbol{\xi}}{\mathrm{D}t}, \qquad \frac{\Delta\rho}{\rho} = \frac{1-\mathcal{J}}{\mathcal{J}}, \quad \mathcal{J} = \det\left(\delta^i_j + \frac{\partial \xi^i}{\partial x_j}\right)$$

Dynamics may be derived from the Lagrangian density:

$$\mathcal{L} = \frac{1}{2}\rho\left|\mathbf{v} + \frac{\partial\boldsymbol{\xi}}{\partial t} + \mathbf{v}\cdot\nabla\boldsymbol{\xi}\right|^2 - p\frac{\mathcal{J}^{1-\gamma}}{\gamma-1} - \rho\Phi(\mathbf{x}+\boldsymbol{\xi}).$$

Perturbative expansion

Expansion of the Lagrangian density, $\mathcal{L}^{(n)} \sim \xi^n$

$$\mathcal{L} = \mathcal{L}^{(0)} + \mathcal{L}^{(1)} + \mathcal{L}^{(2)} + \mathcal{L}^{(3)} + \mathcal{L}^{(4)} + \mathcal{L}^{(5)} + \mathcal{O}(\xi^6),$$

Euler-Lagrange → single governing equation:

$$\begin{aligned}\frac{\mathrm{D}^2\xi_i}{\mathrm{D}t^2} - \frac{1}{\rho}(\gamma - 1)\nabla_i(p\nabla_k\xi^k) - \frac{1}{\rho}\nabla_k(p\nabla_i\xi^k) + \xi^k\nabla_k\nabla_i\Phi = \\ = a_i^{(2)}(\xi) + a_i^{(3)}(\xi) + a_i^{(4)}(\xi)\end{aligned}$$

- LHS → Linear terms
- RHS → Nonlinear accelerations

Note: Papaloizou-Pringle equation

$$\frac{1}{r}\frac{\partial}{\partial r}\left(\rho\frac{\partial\mathcal{W}}{\partial r}\right)+\frac{\partial}{\partial z}\left(\rho\frac{\partial\mathcal{W}}{\partial z}\right)-\frac{m^2}{r^2}\rho\mathcal{W}+\frac{n\rho^2}{(n+1)p}\sigma^2\mathcal{W}=0$$

- Linear perturbations of axisymmetric azimuthal flow
- Perturbation quantity

$$\mathcal{W}\equiv\frac{\delta p}{\rho\sigma},\quad \sigma\equiv\omega-m\Omega$$

- Relation to the Lagrangian displacement

$$\begin{aligned}\xi^r &= -\frac{1}{\sigma^2-\kappa^2}\left(\sigma\frac{\partial\mathcal{W}}{\partial r}-\frac{m\kappa^2}{2\Omega r}\mathcal{W}\right)\\ \xi^\phi &= \frac{i}{\sigma^2-\kappa^2}\left(\frac{2\Omega}{r}\frac{\partial\mathcal{W}}{\partial r}-\frac{m\sigma}{r^2}\mathcal{W}\right)\\ \xi^z &= -\frac{1}{\sigma}\frac{\partial\mathcal{W}}{\partial z}\end{aligned}$$

Linear modes

- Solution of the linear problem:

$$\boldsymbol{\xi}(\mathbf{x}, t) \equiv \boldsymbol{\xi}(\mathbf{x}) \exp\left[\mathrm{i}\omega t\right]$$

- Eigenfunctions $\boldsymbol{\xi}_\alpha(\mathbf{x})$ form (non)orthogonal basis with respect to the scalar product,

$$\langle \xi, \xi' \rangle \equiv \int_V \rho \boldsymbol{\xi} \cdot \boldsymbol{\xi}' \mathrm{d}V$$

- General perturbation (i.e. solution of the nonlinear equation):

$$\boldsymbol{\xi}(\mathbf{x}, t) = \sum_\alpha c_\alpha(t) \boldsymbol{\xi}_\alpha(\mathbf{x}) + \bar{c}_\alpha(t) \bar{\boldsymbol{\xi}}_\alpha(\mathbf{x})$$

Harmonic oscillators

Governing equation → equations for $c_\alpha(t)$

$$\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} + \mathrm{i}\omega_\alpha c_\alpha = \frac{\mathrm{i}}{b_\alpha}\langle \boldsymbol{\xi}_\alpha, \mathbf{a}\rangle$$

where

$$\begin{aligned}
\langle \boldsymbol{\xi}_\alpha, \mathbf{a}^{(2)}\rangle &= \kappa_{\bar\alpha\beta\gamma} c_\beta c_\gamma + \kappa_{\bar\alpha\bar\beta\gamma} \bar c_\beta c_\gamma + \kappa_{\bar\alpha\beta\bar\gamma} c_\beta \bar c_\gamma + \kappa_{\bar\alpha\bar\beta\bar\gamma} \bar c_\beta \bar c_\gamma\,, \\
\langle \boldsymbol{\xi}_\alpha, \mathbf{a}^{(3)}\rangle &= \kappa_{\bar\alpha\beta\gamma\delta} c_\beta c_\gamma c_\delta + \kappa_{\bar\alpha\bar\beta\gamma\delta} \bar c_\beta c_\gamma c_\delta + \kappa_{\bar\alpha\beta\bar\gamma\delta} c_\beta \bar c_\gamma c_\delta + \kappa_{\bar\alpha\beta\gamma\bar\delta} c_\beta c_\gamma \bar c_\delta + {} \\
&\quad \kappa_{\bar\alpha\bar\beta\bar\gamma\delta} \bar c_\beta \bar c_\gamma c_\delta + \kappa_{\bar\alpha\bar\beta\gamma\bar\delta} \bar c_\beta c_\gamma \bar c_\delta + \kappa_{\bar\alpha\beta\bar\gamma\bar\delta} c_\beta \bar c_\gamma \bar c_\delta + \kappa_{\bar\alpha\bar\beta\bar\gamma\bar\delta} \bar c_\beta \bar c_\gamma \bar c_\delta\,, \\
\langle \boldsymbol{\xi}_\alpha, \mathbf{a}^{(4)}\rangle &= \kappa_{\bar\alpha\beta\gamma\delta\epsilon} c_\beta c_\gamma c_\delta c_\epsilon + \kappa_{\bar\alpha\bar\beta\gamma\delta\epsilon} \bar c_\beta c_\gamma c_\delta c_\epsilon + \kappa_{\bar\alpha\beta\bar\gamma\delta\epsilon} c_\beta \bar c_\gamma c_\delta c_\epsilon + \kappa_{\bar\alpha\beta\gamma\bar\delta\epsilon} c_\beta c_\gamma \bar c_\delta c_\epsilon + {} \\
&\quad \kappa_{\bar\alpha\beta\gamma\delta\bar\epsilon} c_\beta c_\gamma c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\bar\beta\bar\gamma\delta\epsilon} \bar c_\beta \bar c_\gamma c_\delta c_\epsilon + \kappa_{\bar\alpha\bar\beta\gamma\bar\delta\epsilon} \bar c_\beta c_\gamma \bar c_\delta c_\epsilon + \kappa_{\bar\alpha\bar\beta\gamma\delta\bar\epsilon} \bar c_\beta c_\gamma c_\delta \bar c_\epsilon + {} \\
&\quad \kappa_{\bar\alpha\beta\bar\gamma\bar\delta\epsilon} c_\beta \bar c_\gamma \bar c_\delta c_\epsilon + \kappa_{\bar\alpha\beta\bar\gamma\delta\bar\epsilon} c_\beta \bar c_\gamma c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\beta\gamma\bar\delta\bar\epsilon} c_\beta c_\gamma \bar c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\bar\beta\bar\gamma\bar\delta\epsilon} \bar c_\beta \bar c_\gamma \bar c_\delta c_\epsilon + {} \\
&\quad \kappa_{\bar\alpha\bar\beta\bar\gamma\delta\bar\epsilon} \bar c_\beta \bar c_\gamma c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\bar\beta\gamma\bar\delta\bar\epsilon} \bar c_\beta c_\gamma \bar c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\beta\bar\gamma\bar\delta\bar\epsilon} c_\beta \bar c_\gamma \bar c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\bar\beta\bar\gamma\bar\delta\bar\epsilon} \bar c_\beta \bar c_\gamma \bar c_\delta \bar c_\epsilon\,.
\end{aligned}$$

...Equations for nonlinear oscillators → Multiple scales.

Coupling coefficients

- Second order → three-modes coupling

$$\begin{aligned}\kappa_{\alpha\beta\gamma} &= \frac{1}{2}\int_V \Big\{ p(\gamma-1)^2 \eta_\alpha \eta_\beta \eta_\gamma + 3p(\gamma-1)\eta_{[\alpha}\eta_{\beta\gamma]} + 2p\eta_{\alpha\beta\gamma} - \\ &\quad \rho\, \xi^i_\alpha \xi^j_\beta \xi^k_\gamma \nabla_i \nabla_j \nabla_k \Phi \Big\}\, \mathrm{d}V,\end{aligned}$$

- Third order → four-modes coupling

$$\begin{aligned}\kappa_{\alpha\beta\gamma\delta} &= -\frac{1}{3!}\int_V \Big\{ \gamma(3-3\gamma+\gamma^2) p\, \eta_\alpha \eta_\beta \eta_\gamma \eta_\delta + 8\gamma p\, \eta_{[\alpha}\eta_{\beta\gamma\delta]} + \\ &\quad 6\gamma(\gamma-2)p\, \eta_{[\alpha}\eta_\beta \eta_{\gamma\delta]} + \rho\, \xi^i_\alpha \xi^j_\beta \xi^k_\gamma \xi^l_\delta \nabla_i \nabla_j \nabla_k \nabla_l \Phi \Big\}\, \mathrm{d}V\end{aligned}$$

- Fifth order → five-modes coupling

$$\begin{aligned}\kappa_{\alpha\beta\gamma\delta\epsilon} &= \frac{1}{4!}\int_V \Big\{ \gamma(1+6\gamma-4\gamma^2+3\gamma^3) p\, \eta_\alpha \eta_\beta \eta_\gamma \eta_\delta \eta_\epsilon + \\ &\quad 10\gamma^2(\gamma-3)p\, \eta_{[\alpha}\eta_\beta \eta_\gamma \eta_{\delta\epsilon]} + 15\gamma(\gamma-1)p\, \eta_{[\alpha}\eta_{\beta\gamma}\eta_{\delta\epsilon]} + 20\gamma^2 p\, \eta_{[\alpha}\eta_\beta \eta_{\gamma\delta\epsilon]} + \\ &\quad 20\gamma p\, \eta_{[\alpha\beta}\eta_{\gamma\delta\epsilon]} - \rho\, \xi^i_\alpha \xi^k_\beta \xi^l_\gamma \xi^m_\delta \xi^n_\epsilon \nabla_i \nabla_k \nabla_l \nabla_m \nabla_n \Phi \Big\}\, \mathrm{d}V,\end{aligned}$$

Symmetry arguments

Symmetry of equilibrium

$$\begin{aligned}(f_\star Q)(x^i) &\equiv \quad Q(-x^i) = Q(x^i),\\ (f_\star Q^i)(x^i) &\equiv -Q^i(-x^i) = Q^i(x^i)\end{aligned}$$

→ modes of defined parity:

$$f_\star \boldsymbol{\xi}_\alpha = \epsilon_\alpha \boldsymbol{\xi}_\alpha, \quad \epsilon_\alpha = \pm 1$$

Coupling coefficient

$$\begin{aligned}\kappa_{\bar\alpha\beta\gamma} &\equiv \langle \bar{\boldsymbol{\xi}}_\alpha, \mathbf{a}^{(2)}(\boldsymbol{\xi}_\beta, \boldsymbol{\xi}_\gamma)\rangle = f_\star \kappa_{\bar\alpha\beta\gamma} = \langle f_\star \bar{\boldsymbol{\xi}}_\alpha, \mathbf{a}^{(2)}(f_\star\boldsymbol{\xi}_\beta, f_\star\boldsymbol{\xi}_\gamma)\rangle = \\ &= \langle \epsilon_\alpha \bar{\boldsymbol{\xi}}_\alpha, \mathbf{a}^{(2)}(\epsilon_\beta\boldsymbol{\xi}_\beta, \epsilon_\gamma\boldsymbol{\xi}_\gamma)\rangle = \epsilon_\alpha\epsilon_\beta\epsilon_\gamma \kappa_{\bar\alpha\beta\gamma}\end{aligned}$$

Then

$$\left(1 - \epsilon_\alpha\epsilon_\beta\epsilon_\gamma\right) \kappa_{\bar\alpha\beta\gamma} = 0$$

→ $\kappa_{\dots}$ vanishes when odd number of odd modes is involved.

A simplification of the governing equations

$$\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} + \mathrm{i}\omega_\alpha c_\alpha = \frac{\mathrm{i}}{b_\alpha}\langle \boldsymbol{\xi}_\alpha, \mathbf{a} \rangle$$

where

$$\begin{aligned}
\langle \boldsymbol{\xi}_\alpha, \mathbf{a}^{(2)} \rangle &= \kappa_{\bar\alpha\beta\gamma} c_\beta c_\gamma + \kappa_{\bar\alpha\bar\beta\gamma} \bar c_\beta c_\gamma + \kappa_{\bar\alpha\beta\bar\gamma} c_\beta \bar c_\gamma + \kappa_{\bar\alpha\bar\beta\bar\gamma} \bar c_\beta \bar c_\gamma , \\
\langle \boldsymbol{\xi}_\alpha, \mathbf{a}^{(3)} \rangle &= \kappa_{\bar\alpha\beta\gamma\delta} c_\beta c_\gamma c_\delta + \kappa_{\bar\alpha\bar\beta\gamma\delta} \bar c_\beta c_\gamma c_\delta + \kappa_{\bar\alpha\beta\bar\gamma\delta} c_\beta \bar c_\gamma c_\delta + \kappa_{\bar\alpha\beta\gamma\bar\delta} c_\beta c_\gamma \bar c_\delta + \\
&\quad \kappa_{\bar\alpha\bar\beta\bar\gamma\delta} \bar c_\beta \bar c_\gamma c_\delta + \kappa_{\bar\alpha\bar\beta\gamma\bar\delta} \bar c_\beta c_\gamma \bar c_\delta + \kappa_{\bar\alpha\beta\bar\gamma\bar\delta} c_\beta \bar c_\gamma \bar c_\delta + \kappa_{\bar\alpha\bar\beta\bar\gamma\bar\delta} \bar c_\beta \bar c_\gamma \bar c_\delta , \\
\langle \boldsymbol{\xi}_\alpha, \mathbf{a}^{(4)} \rangle &= \kappa_{\bar\alpha\beta\gamma\delta\epsilon} c_\beta c_\gamma c_\delta c_\epsilon + \kappa_{\bar\alpha\bar\beta\gamma\delta\epsilon} \bar c_\beta c_\gamma c_\delta c_\epsilon + \kappa_{\bar\alpha\beta\bar\gamma\delta\epsilon} c_\beta \bar c_\gamma c_\delta c_\epsilon + \kappa_{\bar\alpha\beta\gamma\bar\delta\epsilon} c_\beta c_\gamma \bar c_\delta c_\epsilon + \\
&\quad \kappa_{\bar\alpha\beta\gamma\delta\bar\epsilon} c_\beta c_\gamma c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\bar\beta\bar\gamma\delta\epsilon} \bar c_\beta \bar c_\gamma c_\delta c_\epsilon + \kappa_{\bar\alpha\bar\beta\gamma\bar\delta\epsilon} \bar c_\beta c_\gamma \bar c_\delta c_\epsilon + \kappa_{\bar\alpha\bar\beta\gamma\delta\bar\epsilon} \bar c_\beta c_\gamma c_\delta \bar c_\epsilon + \\
&\quad \kappa_{\bar\alpha\beta\bar\gamma\bar\delta\epsilon} c_\beta \bar c_\gamma \bar c_\delta c_\epsilon + \kappa_{\bar\alpha\beta\bar\gamma\delta\bar\epsilon} c_\beta \bar c_\gamma c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\beta\gamma\bar\delta\bar\epsilon} c_\beta c_\gamma \bar c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\bar\beta\bar\gamma\bar\delta\epsilon} \bar c_\beta \bar c_\gamma \bar c_\delta c_\epsilon + \\
&\quad \kappa_{\bar\alpha\bar\beta\bar\gamma\delta\bar\epsilon} \bar c_\beta \bar c_\gamma c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\bar\beta\gamma\bar\delta\bar\epsilon} \bar c_\beta c_\gamma \bar c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\beta\bar\gamma\bar\delta\bar\epsilon} c_\beta \bar c_\gamma \bar c_\delta \bar c_\epsilon + \kappa_{\bar\alpha\bar\beta\bar\gamma\bar\delta\bar\epsilon} \bar c_\beta \bar c_\gamma \bar c_\delta \bar c_\epsilon .
\end{aligned}$$

... Not all $\kappa_{\dots}$ present!

Two-modes internal resonances

Resonance	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
4 : 1	YES	—	YES	—
3 : 1	YES	—	—	**YES**
2 : 1	**YES**	—	**YES**	—
3 : 2	YES	**YES**	—	—
1 : 1	YES	$u_{\alpha\alpha} = 0$ $u_{\beta\beta} = 0$ $v_{ij} = 0$	$u_{\alpha\alpha} = 0$ $u_{\beta\beta} = 0$ $u_{ij} = 0$	YES

The possible resonances up to the fourth order according to parities of the two involved modes α and β. When both parities are $+$ all resonances are possible. For modes with different parities the strongest possible resonances are 3:2 or 2:1. The strongest resonances with $\omega_\alpha < \omega_\beta$ are denoted by bold letters.

Harmonic content

Harmonic	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
ω_α	α	α	α	α
ω_β	β	β	β	β
$2\omega_\alpha$	α,β	α	β	—
$\omega_\alpha + \omega_\beta$	α,β	β	α	—
$\omega_\alpha - \omega_\beta$	α,β	β	α	—
$2\omega_\beta$	α,β	α	β	—

Harmonics present in the power-spectra of oscillations up to the second order. Listed are modes whose power spectrum contains given harmonic.

Slender tori

Example system: slender torus

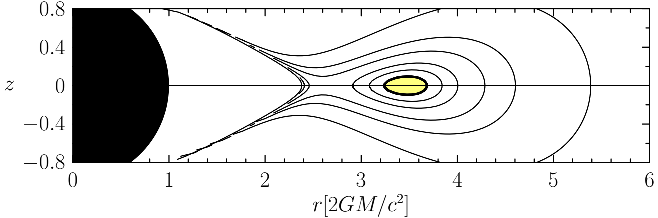
- Polytropic equation of state → $\Omega = \Omega(r)$ only

$$\rho = \rho_0 f^n(r,z), \quad p = p_0 f^{n+1}(r,z)$$

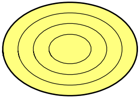
  coincide with constant effective potential surfaces when $\ell \equiv \mathrm{const}$

- the function f is expanded in the vicinity of maximal pressure point:

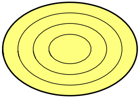
$$f = 1 - \left[\bar{\omega}_r^2 - \frac{2r_0}{\ell_0}\left(\frac{\mathrm{d}\ell}{\mathrm{d}r}\right)_0\right]\bar{x}^2 - \bar{\omega}_z^2\bar{y}_0^2$$



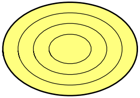
Lowest-order modes (Blaes et al, 2006)



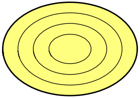
Radial epicyclic



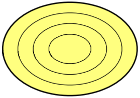
Vertical epicyclic



X-mode



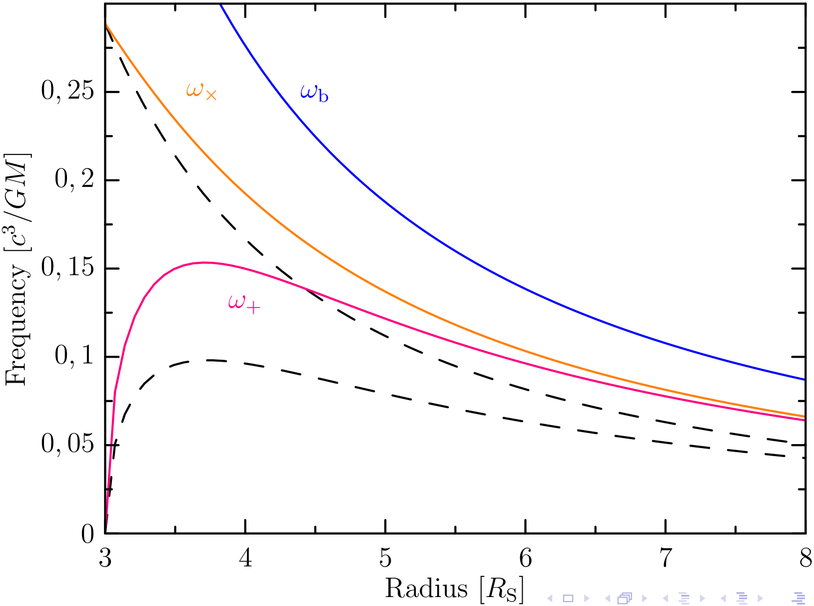
+-mode



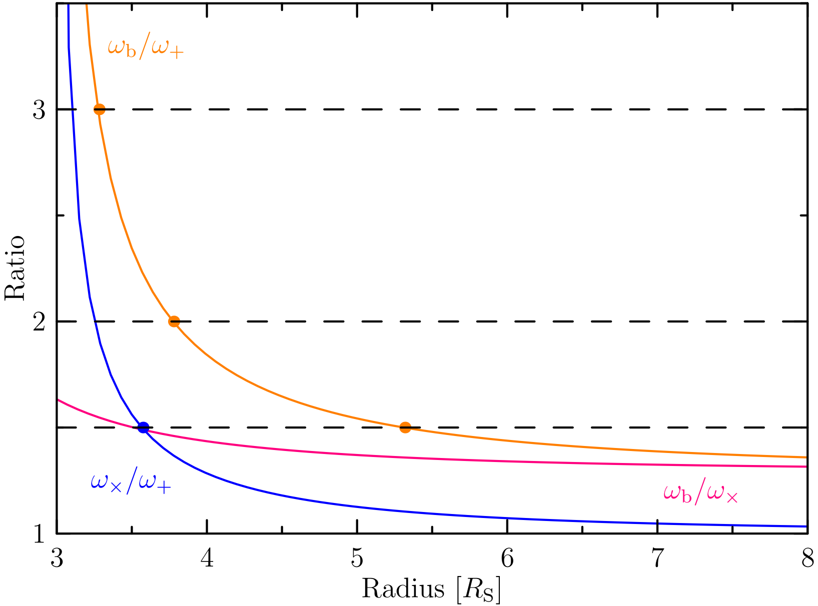
Breathing mode

Eigenfrequencies...

Polytropic index: $n = 3/2$



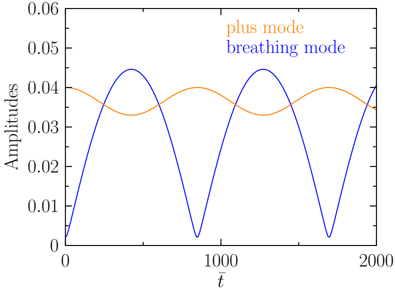
...and their ratios

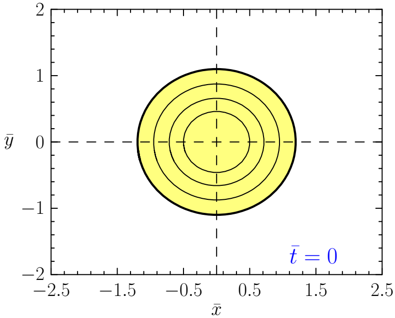


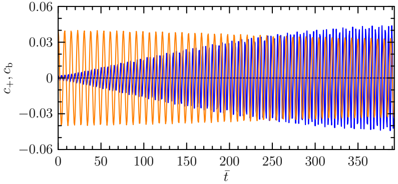
Example of internal resonance

2:1 resonance

$$\frac{\omega_{\mathrm{b}}}{\omega_{+}} = \frac{2}{1}$$







- Exchange of energy
- Low-frequency modulation

Excitation by external forcing

- Perturbation on the star → nonstationary, nonaxisymmetric

$$\Phi(r,z) + \varphi(r,z,\phi - \omega_\star t) = \Phi(r,z) + \sum_{\mu \geq 1} \varphi_\mu(r,z) \cos\left[\mu(\phi - \omega_\star t)\right]$$

- The same order of the excitation and response, $\varphi \sim \xi$

- Governing equation (up to the second order)

$$\frac{\mathrm{d}^2 \xi_i}{\mathrm{d}t^2} - \frac{1}{\rho}(\gamma - 1)\nabla_i(p\nabla_k \xi^k) - \frac{1}{\rho}\nabla_k(p\nabla_i \xi^k) + \xi^k \nabla_k \nabla_i \Phi =$$

$$= k_i^{(1)}(t) + k_i^{(2)}(t, \boldsymbol{\xi}) + a_i^{(2)}(\boldsymbol{\xi})$$

Forcing terms:

$$k_i^{(1)} = -\nabla_i \varphi, \quad k_i^{(2)} = \xi^k \nabla_i \nabla_k \varphi$$

Nonlinear forced oscillators

Equations for modal coefficients

$$\begin{aligned}\frac{\mathrm{d}c_\alpha}{\mathrm{d}t} + \mathrm{i}\,\omega_\alpha c_\alpha \;=\; & \frac{\mathrm{i}}{b_\alpha}\sum_{\beta,\gamma} \lambda_{\bar\alpha}(t) + \lambda_{\bar\alpha\beta}(t) c_\beta + \lambda_{\bar\alpha\bar\beta}(t)\bar c_\beta + \\ & \kappa_{\bar\alpha\beta\gamma} c_\beta c_\gamma + \kappa_{\bar\alpha\bar\beta\gamma}\bar c_\beta c_\gamma + \kappa_{\bar\alpha\beta\bar\gamma} c_\beta \bar c_\gamma + \kappa_{\bar\alpha\bar\beta\bar\gamma}\bar c_\beta \bar c_\gamma,\end{aligned}$$

Forcing coefficients

$$\begin{aligned}\lambda_\alpha(t) \;&=\; -2\pi\cos[m_\alpha\omega_\star t]\int_S \rho\,\xi^i_\alpha \nabla_i(\varphi_{m_\alpha})\,\mathrm{d}S,\\ \lambda_{\alpha\beta}(t) \;&=\; -2\pi\cos[(m_\alpha + m_\beta)\omega_\star t]\int_S \rho\,\xi^i_\alpha \xi^j_\beta \nabla_i\nabla_j(\varphi_{m_\alpha+m_\beta})\,\mathrm{d}S,\end{aligned}$$

Slender torus case

- The forcing coefficient λ_α depends on two integrals

$$\varphi_{m_\alpha r} \int f^n \frac{\partial \mathcal{W}}{\partial \bar{x}} \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y} \quad \text{and} \quad \varphi_{m_\alpha z} \int f^n \frac{\partial \mathcal{W}}{\partial \bar{y}} \, \mathrm{d}\bar{x} \, \mathrm{d}\bar{y}$$

  ...forcing in the radial and vertical direction

- Nonzero contributions only from modes with parities

$$(-,+) \quad \text{and} \quad (+,-)$$

- The only lowest-order modes excited are epicyclic.
- However there are also some higher-order modes, e.g.

$$\begin{aligned} \mathcal{W}^{(-,+)} &= W_{10}\bar{x} + W_{30}\bar{x}^3 + W_{12}\bar{x}^2\bar{y}, \\ \mathcal{W}^{(+,-)} &= W_{01}\bar{y} + W_{21}\bar{x}^2\bar{y} + W_{03}\bar{y}^3, \end{aligned}$$

Conclusions

Conclusions

- Straight-forward method for calculating nonlinear oscillations.
- Key problem: Complete set of linear modes (easy in slender tori, accretion disk??)
- Spectra of resonances in symmetric cases.
- Energy flow between modes in resonances → low-frequency modulation (Horák et al, 2003).

- Free oscillations: Epicyclic modes are independent.
- External excitation → epicyclic modes (only?)
- Analysis of larger tori needed: relation between the two expansion parameters ϵ and β, torus instabilities

Conclusions II

- Mode identification
- Modulation of the accretion rate
- Nonlinear interaction of the oscillations and turbulence
- Radiative processes – elmag. spectra,

...etc