Signatures of a nonlinear resonance in kilo-Hertz QPOs

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16.3.2006

Introduction: Observations

Sources:

- High-frequency QPOs are observed in X-rays of Low Mass X-ray binaries
- observed in both black-hole and neutron star binaries
- Frequencies are comparable with Keplerian frequencies close to the compact object

$$\nu_{\rm ISCO} = 1580 \frac{M_\odot}{M} \, \mathrm{Hz}$$

Frequency pairs (ν_ℓ and ν_u)

- Frequencies scales inversely with the mass of the compact object
- Frequencies are *stable* in black hole sources
- Rational (3:2) ratio

Introduction: Abramowicz and Kluzniak resonance model

1. What oscillates? – frequency identification:
   - Upper QPO frequency == Vertical epicyclic frequency
   - Lower QPO frequency == Radial epicyclic frequency
2. Nonlinear interaction of epicyclic oscillations → resonance:

$$\delta\ddot{r} + \omega_r^2 \delta r = f_r(\delta r, \delta\theta) \quad \delta\ddot{\theta} + \omega_\theta^2 \delta\theta = f_\theta(\delta r, \delta\theta)$$

Results: Expanding up to the second order in vertical equations,

$$\delta\ddot{\theta} + \omega_\theta^2 (1 + k\delta r)\delta\theta = 0$$

and assuming small radial oscillations, $\delta r \propto \cos(\omega_r t)$, we get Mathieu equation. Parametric resonance (exponential grow of $\delta\theta(t)$) when

$$\frac{\omega_\theta}{\omega_r} = \frac{n}{2}$$

GR$\rightarrow$ $\omega_\theta \geq \omega_r \rightarrow n = 3$ is the strongest resonance

General effects of nonlinear coupling

System of two coupled oscillators

$$\begin{aligned} \ddot{x}_\ell + \omega_\ell^2 x_\ell &= f_\ell(x_\ell, x_u, \dot{x}_\ell, \dot{x}_u) \\ \ddot{x}_u + \omega_u^2 x_u &= f_u(x_\ell, x_u, \dot{x}_\ell, \dot{x}_u) \end{aligned}$$

Conservation of the energy (time-reflection invariance)

$$f(x_\ell, x_u, -\dot{x}_\ell, -\dot{x}_u) = f(x_\ell, x_u, \dot{x}_\ell, \dot{x}_u)$$

Equatorial-plane reflection invariance

$$f(x_\ell, -x_u, \dot{x}_\ell, -\dot{x}_u) = f(x_\ell, x_u, \dot{x}_\ell, \dot{x}_u)$$

Resonance region

Possible resonances up to the fourth order

$$\frac{\omega_u}{\omega_\ell} = \frac{1}{2},\ 1,\ \frac{3}{2},\ \frac{1}{4}$$

- The strongest possible resonance if $\omega_u > \omega_\ell$is 3 : 2 as in the case of the Mathieu equation.
- New resonance 1 : 4 appears in general.

Character of the resonance 3:2

Small oscillations (weak coupling): The solution of the form

$$x_\ell = A_\ell \mathrm{e}^{i\omega_\ell t} + \mathrm{cc} \quad A_\ell = \tfrac{1}{2} a_\ell e^{i\phi_\ell}$$

The amplitudes a_ℓ and a_u are governed by

$$\dot{a}_\ell = \tfrac{1}{16}\beta\nu a_\ell^2 a_u^2 \sin\gamma, \quad \dot{a}_u = -\tfrac{1}{16}\beta a_\ell^3 a_u \sin\gamma$$

and the phase function $\gamma \equiv 2\phi_u - 3\phi_\ell - \sigma t$

$$\dot{\gamma} = -\sigma + \tfrac{1}{4}\left[\mu_\ell a_\ell^2 + \mu_u a_u^2 + \tfrac{1}{2}\beta a_\ell \left(-a_\ell^2 + \tfrac{3}{2}\nu a_u^2\right)\cos\gamma\right]$$

Parameters of the system:

$$\mu_\ell = \mu_u = \beta = 1, \quad \nu \approx \left(\frac{3}{2}\right)^2$$

Detuning parameter:

$$\sigma \equiv -\frac{2}{R}\left(R - \frac{3}{2}\right), \quad R \equiv \frac{\omega_u}{\omega_\ell}$$

Dynamics close to the resonance II.

Conservation of the energy: $a_\ell^2 + \nu a_u^2 = E$

$$a_\ell^2 = \xi E, \quad a_u^2 = \frac{1}{\nu}(1-\xi)E$$

The evolution of the system is by

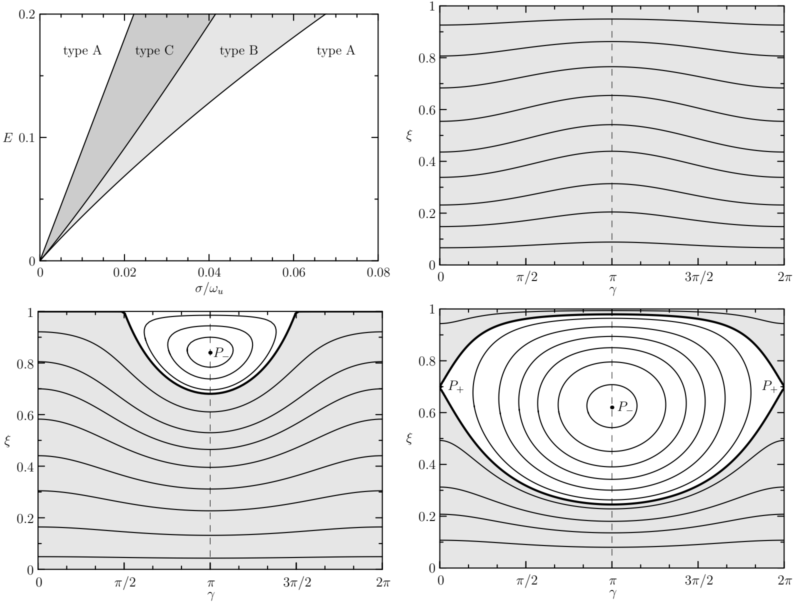
$$\dot{\xi} = \tfrac{1}{8}\beta(E\xi)^{3/2}(1-\xi)\sin\gamma$$

and

$$\dot{\gamma} = -\sigma + \tfrac{1}{4}E\left[\mu_\ell\xi + \frac{\mu_u}{\nu}(1-\xi) + \tfrac{1}{4}\beta(E\xi)^{1/2}(3-5\xi)\cos\gamma\right]$$

Critical points: $\dot{\gamma} \equiv \dot{\xi} \equiv 0$

Resonance tongues and topologies



Two notes on observed frequencies

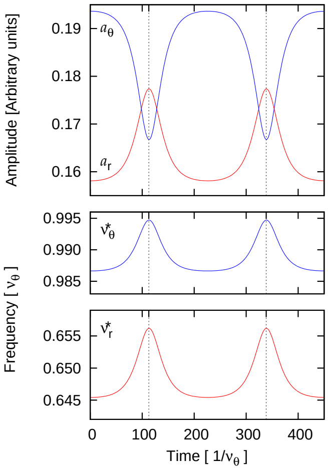
1. The observed frequencies are in exact 3:2 whenever $\dot{\gamma} = 0$:

$$\begin{aligned} \dot{\gamma} &= 2\dot{\phi}_u - 3\dot{\phi}_\ell - \sigma \\ &= 2\dot{\phi}_u - 3\dot{\phi}_\ell - (3\omega_\ell - 2\omega_u) = 2\omega_u^\star - 3\omega_\ell^\star \end{aligned}$$

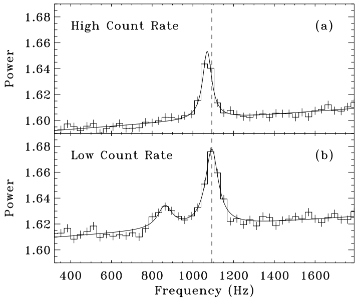
2. The corrections to the eigenfrequencies:

$$\begin{aligned} \Delta\omega_\ell &= \dot{\phi}_\ell = -\frac{1}{2}\left[\kappa_\ell a_\ell^2 + \kappa_u a_u^2\right] - \frac{\beta\nu}{16} a_\ell a_u^2 \cos\gamma \\ \Delta\omega_u &= \dot{\phi}_\ell = -\frac{1}{3}\left[\lambda_\ell a_\ell^2 + \lambda_u a_u^2\right] - \frac{\beta}{16} a_\ell^3 \cos\gamma \end{aligned}$$

Low-frequency modulation



$$T \sim \frac{16\pi}{\beta\omega_\theta}\,\mathcal{E}^{-3/2}$$



Observed in several QPO sources (Sco X-1, XTE 1550-564)

Effects of slow change in parameters

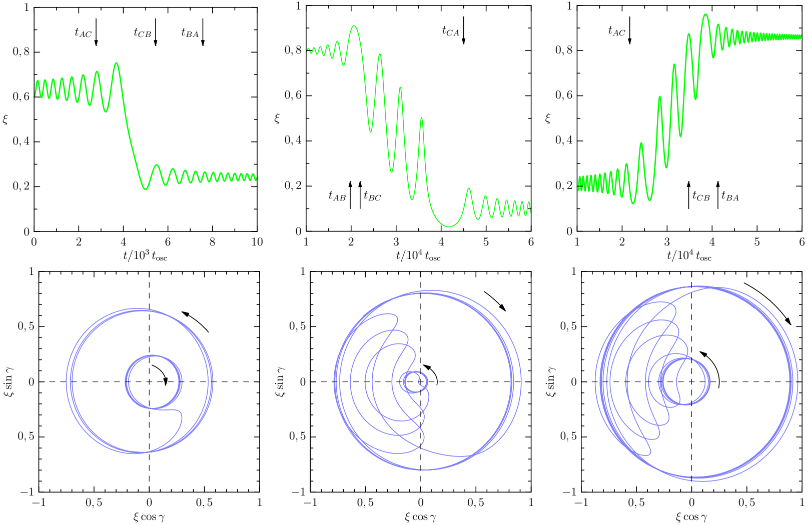
The position in the (σ, E) plane is slowly changed:

$$\sigma = \sigma(t), \quad E = E(t), \quad t_{\rm char} \ll \frac{1}{\omega_u}$$

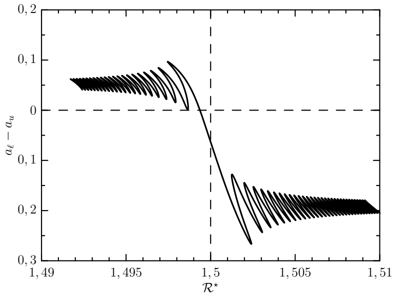
Motion or expansion of the accretion torus, energy inflow,...

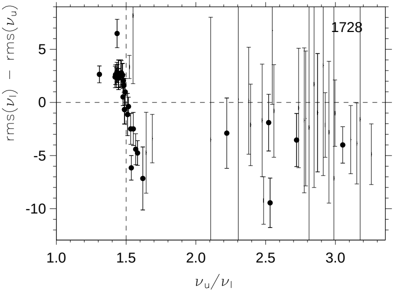
- Change of the topology
- Jump in ξ

Effects of slow change in parameters



Is this observed in real systems?





Discussion and conclusions – I.

- Jump in the amplitudes as the system pass the resonance tongues in the (σ, E) plane.
- The observed frequencies are in *exact* 3:2 ratio.

However (TBE)

- RMS $\rightarrow a_\ell$, a_u? (modulation mechanism?)
- Dependence on the initial contitions (i.e. γ_0)?
- Non-conservative systems (limit cycles, ...)
- Which frequencies are more fundamental?
- Nature of the coupling (PHYSICS)

Could this be observed?