Nonlinear interaction among oscillation modes of accretion tori

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Outline

- Nonlinear interaction of modes
  → How to calculate nonlinear oscillations of fluid bodies?
- Slender tori
  → Eigenfrequencies and eigenfunctions
  → Two-mode coupling, internal resonances
  → Epicyclic resonance
- Excitation of the oscillations by an intability
  → Papaloizou-Pringle instability
  → Three-mode interctions
- Conclusions, open quastions

Nonlinear interactions of modes

Perturbative approach to nonlinearities

Governing equation ($\boldsymbol{\xi} =$ Lagrangian displacement):

$$\boxed{\frac{\mathrm{D}^2\xi_i}{\mathrm{D}t^2} - \frac{1}{\rho}\nabla_j\left[(\gamma-1)p(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})g^{ij} + p\nabla^i\xi^j\right] + \xi^k\nabla_k\nabla_i\Phi = \sum_n a_i^{(n)}(\boldsymbol{\xi})}$$

- RHS → Nonlinear accelerations (*perturbation*)
- Linear equation ⇒ eigenmodes $\{\omega_A, \boldsymbol{\xi}_A\}$,
- $\{\boldsymbol{\xi}_A\}$ is *complete* → Solution of nonlinear equation

$$\boldsymbol{\xi}(\mathbf{x}, t) = \sum_A c_A(t)\boldsymbol{\xi}_A(\mathbf{x})$$

The equation governing nonlinear oscillations

$$\boxed{\frac{\mathrm{d}c_A}{\mathrm{d}t} + \mathrm{i}\omega_A c_A = \frac{\mathrm{i}}{b_A}\mathcal{F}_A(c_I)}$$

… coupled oscillators

Nonlinear coupling functions

$$\mathcal{F}_A(c_I) = \sum_{B,C} \kappa_{ABC}\, c_B c_C + \sum_{B,C,D} \kappa_{ABCD}\, c_B c_C c_D + \dots$$

- Second order $\longrightarrow$ three-modes coupling

$$\begin{aligned}\kappa_{ABC} &= \frac{1}{2}\int_V \Big\{ p(\gamma-1)^2 \eta_A\eta_B\eta_C + 3p(\gamma-1)\eta_{[A}\eta_{BC]} + 2p\eta_{ABC} - \\ &\quad \rho\, \xi^i_A \xi^j_B \xi^k_C \nabla_i\nabla_j\nabla_k \Phi \Big\}\, \mathrm{d}V,\end{aligned}$$

- Third order $\longrightarrow$ four-modes coupling

$$\begin{aligned}\kappa_{ABCD} &= -\frac{1}{3!}\int_V \Big\{ \gamma(3-3\gamma+\gamma^2) p\, \eta_A\eta_B\eta_C\eta_D + 8\gamma p\, \eta_{[A}\eta_{BCD]} + \\ &\quad 6\gamma(\gamma-2)p\, \eta_{[A}\eta_B\eta_{CD]} + \rho\, \xi^i_A\xi^j_B\xi^k_C\xi^l_D \nabla_i\nabla_j\nabla_k\nabla_l\Phi \Big\}\, \mathrm{d}V\end{aligned}$$

- Fifth order $\longrightarrow$ five-modes coupling

$$\begin{aligned}\kappa_{ABCDE} &= \frac{1}{4!}\int_V \Big\{ \gamma(1+6\gamma-4\gamma^2+3\gamma^3) p\, \eta_A\eta_B\eta_C\eta_D\eta_E + \\ &\quad 10\gamma^2(\gamma-3) p\, \eta_{[A}\eta_B\eta_C\eta_{DE]} + 15\gamma(\gamma-1) p\, \eta_{[A}\eta_{BC}\eta_{DE]} + 20\gamma^2 p\, \eta_{[A}\eta_B\eta_{CDE]} + \\ &\quad 20\gamma p\, \eta_{[AB}\eta_{CDE]} - \rho\, \xi^i_A\xi^k_B\xi^l_C\xi^m_D\xi^n_E \nabla_i\nabla_k\nabla_l\nabla_m\nabla_n\Phi \Big\}\, \mathrm{d}V,\end{aligned}$$

Slender tori

Example system: slender torus

- Polytropic equation of state:

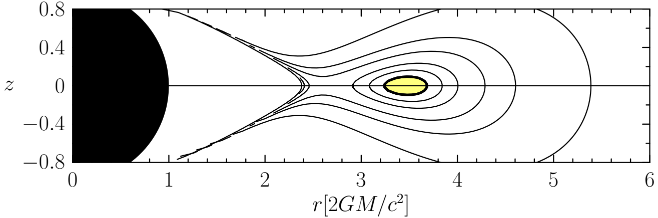
$$\rho = \rho_0 f^n(r,z), \quad p = p_0 f^{n+1}(r,z)$$

- Small filling parameter:

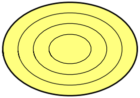
$$\beta^2 = 2(n+1)p_0/(\rho_0 r_0^2 \Omega_0^2) \sim (\Delta r/r_0)^2 \ll 1$$

- the function f is expanded in the maximal pressure point:

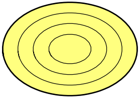
$$\boxed{f = 1 - \left[\bar{\omega}_r^2 - \frac{2r_0}{\ell_0}\left(\frac{\mathrm{d}\ell}{\mathrm{d}r}\right)_0\right]\bar{x}^2 - \bar{\omega}_z^2\bar{y}_0^2} \quad \text{...ellipses}$$



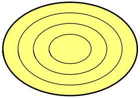
Lowest-order modes (Blaes et al, 2006)



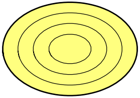
Radial epicyclic



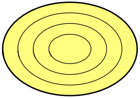
Vertical epicyclic



X-mode



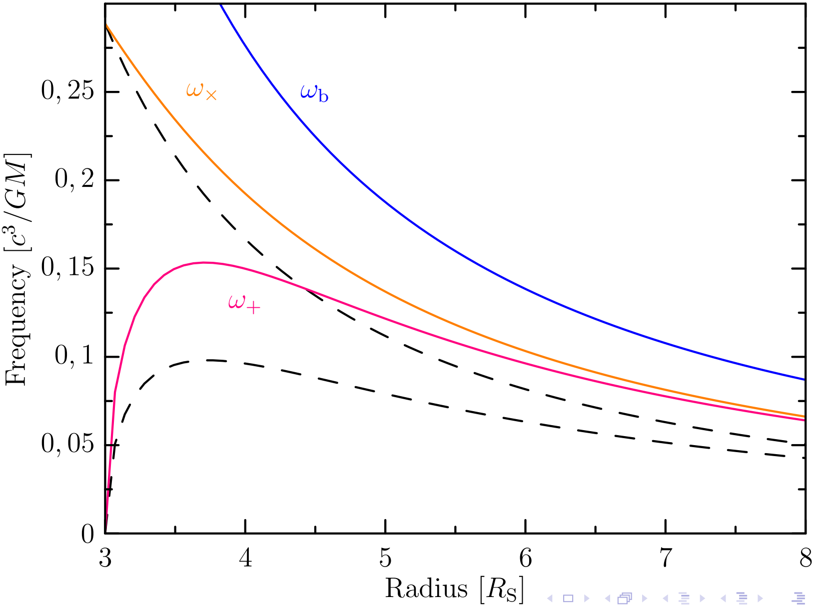
$+$-mode



Breathing mode

Eigenfrequencies...

Polytropic index: $n = 3/2$



Two-mode interaction: internal resonances

- $\omega_A/\omega_B = n/m \Rightarrow$ Internal resonance

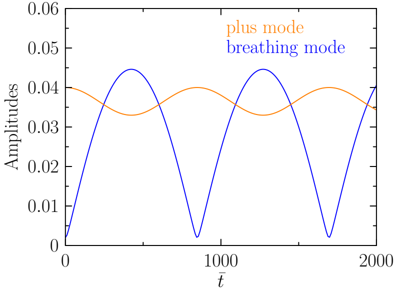
- Equatorial-plane symmetry of the system
  1. $\boldsymbol{\xi}_A$ are either even or odd.
  2. $\kappa_{AB\ldots} = 0$ if odd number of odd modes are involded
  3. Some resonances are absent...

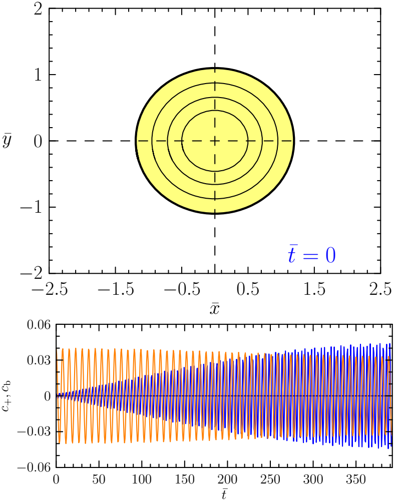
Resonance	Order	$(+,+)$	$(+,-)$	$(-,+)$	$(-,-)$
$2:1$	second	YES	—	YES	—
$3:1$	third	YES	—	—	YES
$4:1$	fourth	YES	—	YES	—
$3:2$	fourth	YES	YES	—	—

Example of the internal resonance

2:1 resonance

$$\frac{\omega_{\mathrm{b}}}{\omega_+} = \frac{2}{1}$$





- Exchange of energy
- Low-frequency modulation

Epicyclic resonance

General k-mode coupling coefficient among epicyclic modes:

$$\kappa_{[k]}(\boldsymbol{\xi}_A, \boldsymbol{\nabla}\boldsymbol{\xi}_B) \equiv \kappa_{[k]}^{(\mathrm{p})}(\boldsymbol{\nabla}\boldsymbol{\xi}_A) + \kappa_{[k]}^{(\mathrm{g})}(\boldsymbol{\xi}_B)$$

- Nodal modes: $\boxed{\kappa_{[k]}^{(\mathrm{p})}/(b\Omega_0) \sim 1,}$ $\kappa_{[k]}^{(\mathrm{g})}/(b\Omega_0) \sim \beta^{k-2}$
- Epicyclic modes: $\kappa_{[k]}^{(\mathrm{p})}/(b\Omega_0) \sim \beta^{k},$ $\boxed{\kappa_{[k]}^{(\mathrm{g})}/(b\Omega_0) \sim \beta^{k-2}}$

Estimates for epicyclic resonance ($\epsilon \sim \xi/\Delta r$):

- Characteristic timescale: $T_{\mathrm{mod}} \sim \beta^{-3}\epsilon^{-3}\Omega_0^{-1}$
- Resonance range: $\Delta\nu/\Omega_0 \sim \beta^2\epsilon^2$

Comparison with other timescales of the torus ($T_{\mathrm{therm}}, T_{\mathrm{visc}}$):

Resonance is possible when $\boxed{\alpha_{\mathrm{t}} \lesssim \epsilon^5\beta^3}$

Resonance + turbulence

Stochastic excitation

$$\frac{\mathrm{d}c_A}{\mathrm{d}t} + \mathrm{i}\omega_A c_A = \frac{\mathrm{i}}{b_A}\left[\mathcal{F}_A(c_I) + Q_A(t)\right]$$

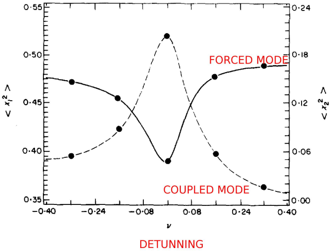
→ $Q_A(t)$ is a stochastic function

Inspiration:

Nayfeh & Serhan (1989):

→ 1:2 internal resonance

→ coupled oscillation excited



Three-mode resonances

and

nonlinear Papaloizou-pringle instability

Papaloizou-Pringle instability in slender tori

- Constant angular momentum distribution
- Expansion in the torus thickness [Blaes & Šrámková]:

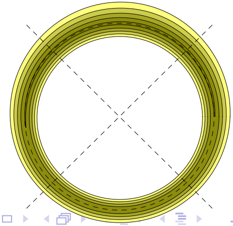
$$\boxed{\beta \equiv \frac{\Delta r}{R}} \qquad \omega = \omega^{(0)} + \beta\omega^{(1)} + \dots, \qquad W = \frac{\delta p}{\rho\sigma} = W^{(0)} + \beta W^{(1)} + \dots .$$

Corotation mode: Marginal stability

$$\begin{aligned} \omega_0 &= \Omega_0 + \boxed{\mathrm{i}\sqrt{2}\, m\beta\, b} \\ W_0 &= C_0 \left\{ 1 + m^2\beta^2 \left[a^2\bar{x}^2 - b^2\bar{y}^2 + \frac{4\sqrt{2}\,\mathrm{i}b}{\bar{\omega}_r^2}\bar{x} + \frac{\bar{\omega}_r^2 b^2 - \bar{\omega}_z^2 a^2}{2(n+1)\bar{\omega}_r^2\bar{\omega}_z^2} \right] + \mathcal{O}(\beta^3) \right\} \end{aligned}$$

where

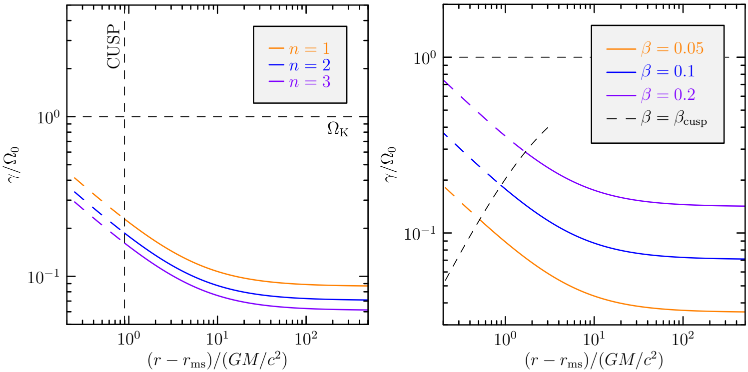
$$a^2 \equiv \frac{4(1+2n) + \bar{\omega}_r^2}{4(1+n)\bar{\omega}_r^2}, \qquad b^2 \equiv \frac{4 - \bar{\omega}_r^2}{4(1+n)\bar{\omega}_r^2}.$$



⇒ Principal mode of the Papaloizou-Pringle

Growth-rates of the unstable mode

Dependence on the polytropic index and torus thickness



Nonlinear evolution: three-mode coupling

- Saturation by resonant interactions with damped modes
- Stars [Dziembowski 82, Moskalik 85, Nowakowski 05,...]

Common resonant triples ($m = 1$ corotation mode):

$$\begin{aligned}\delta\omega \equiv \omega_1 + \omega_2 - \omega_3 &\approx 0 = \omega + (\Omega_0 - \omega) - \Omega_0 + \mathcal{O}(\beta^2)\\ m_1 + m_2 - m_3 &= 0 = 0 + 1 - 1\end{aligned}$$

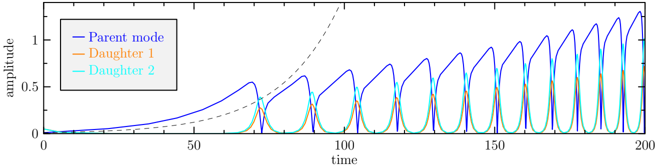
Amplitude equations

$$\begin{aligned}\dot{A}_1 &= \gamma_1 A_1 + \mathrm{i}\,\omega_1 \kappa A_2^* A_3 \mathrm{e}^{\mathrm{i}\delta\omega t}\\ \dot{A}_2 &= \gamma_2 A_2 + \mathrm{i}\,\omega_2 \kappa A_1^* A_3 \mathrm{e}^{\mathrm{i}\delta\omega t}\\ \dot{A}_3 &= \gamma_3 A_3 + \mathrm{i}\,\omega_3 \kappa A_1 A_2 \mathrm{e}^{-\mathrm{i}\delta\omega t}\end{aligned}$$

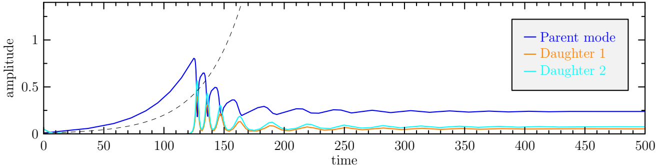
⇒ Unstable 'parent' mode ($\gamma_3 > 0$) → Damped 'daughter' modes ($\gamma_{1,2} < 0$).

Three-mode dynamics

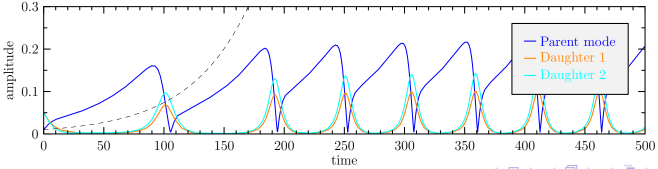
- $|\gamma_3| \gtrsim |\gamma_1| + |\gamma_2| \Rightarrow$ unstable



- $|\gamma_3| \ll |\gamma_1| + |\gamma_2|$ & $\delta\omega > \delta\omega_{\text{crit}} \Rightarrow$ saturation



- $|\gamma_3| \ll |\gamma_1| + |\gamma_2|$ & $\delta\omega < \delta\omega_{\text{crit}} \Rightarrow$ stable limit cycles



Observable consequence?

Three mode coupling condition: $\omega_1 + \omega_2 \approx \omega_3$

→ XTE 1550-564:

$$92\mathrm{Hz} + 184\mathrm{Hz} = 276\mathrm{Hz}$$

→ GRS 1915+105: Fibonacci series (W.K.)

$$16\mathrm{Hz} + 41\mathrm{Hz} \approx 67\mathrm{Hz}$$

$$41\mathrm{Hz} + 67\mathrm{Hz} \approx 113\mathrm{Hz}$$

$$\ldots$$

... ‘Grand-daughter’ modes (?)

Conclusions

Conclusions

- Unstable modes may be saturated by resonant processes
- Damped modes may reach substantial amplitudes
- Limit cycles: Low-frequency modulation (time scale $\propto 1/\gamma_3$)

Main question = damping

- α-viscosity seems to be insuficient
- MHD turbulence, disipation
- accretion
  (reduces excitation rate and increases damping rates)

Other quastions

- Role of the additional internal resonances (e.g. $1 : \mathbf{2} : \mathbf{3}$)
- Saturation of other global instabilities.
  → MRI [linear analysis by Curry & Pudritz 95]