Nonlinear interaction among oscillation modes of accretion tori

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Outline

- Nonlinear interaction of modes
  $\longrightarrow$ How to calculate nonlinear oscillations of fluid bodies?
- Slender tori
  $\longrightarrow$ Eigenfrequencies and eigenfunctions
  $\longrightarrow$ Coupling two modes, internal resonances
- The 3:2 epicyclic resonance
  $\longrightarrow$ Resonance conditions
  $\longrightarrow$ Region of resonance
- Excitation of the oscillations by an instability
  $\longrightarrow$ Papaloizou-Pringle instability
  $\longrightarrow$ Three-mode interctions
- Nonlinear diskoseismology (questions)...?
- Conclusions, open quastions

Nonlinear interactions of modes

Perturbative approach to nonlinearities

Governing equation ($\boldsymbol{\xi} =$ Lagrangian displacement):

$$\boxed{\frac{\mathrm{D}^2\xi_i}{\mathrm{D}t^2} - \frac{1}{\rho}\nabla_j\left[(\gamma-1)p(\boldsymbol{\nabla}\cdot\boldsymbol{\xi})g^{ij} + p\nabla^i\xi^j\right] + \xi^k\nabla_k\nabla_i\Phi = \sum_n a_i^{(n)}(\boldsymbol{\xi})}$$

- RHS $\rightarrow$ Nonlinear accelerations (*perturbation*)
- Linear equation $\Rightarrow$ eigenmodes $\{\omega_A, \boldsymbol{\xi}_A\}$ [Shutz(1980)],

$$\partial_t^2\boldsymbol{\xi} + \hat{B}\partial_t\boldsymbol{\xi} + \hat{C}\boldsymbol{\xi} = 0$$

  $\hat{B}$ is anti-Hermitian, $\hat{C}$ is Hermitian $\rightarrow$ $\{\boldsymbol{\xi}_A\}$ is *still complete*
- Solution of nonlinear equation

$$\boldsymbol{\xi}(\mathbf{x}, t) = \sum_A c_A(t)\boldsymbol{\xi}_A(\mathbf{x})$$

The equation governing nonlinear oscillations

$$\boxed{\frac{\mathrm{d}c_A}{\mathrm{d}t} + \mathrm{i}\omega_A c_A = \frac{\mathrm{i}}{b_A}\mathcal{F}_A(c_I)} \quad \ldots \text{ coupled oscillators}$$

Nonlinear coupling functions

$$\mathcal{F}_A(c_I) = \sum_{B,C} \kappa_{ABC}\, c_B c_C + \sum_{B,C,D} \kappa_{ABCD}\, c_B c_C c_D + \dots$$

- Second order → three-modes coupling [Dziembowski(1982)]

$$\kappa_{ABC} = \frac{1}{2}\int_V \Big\{ p(\gamma-1)^2 \eta_A\eta_B\eta_C + 3p(\gamma-1)\eta_{[A}\eta_{BC]} + 2p\eta_{ABC} - \rho\, \xi_A^i \xi_B^j \xi_C^k \nabla_i\nabla_j\nabla_k \Phi \Big\}\, \mathrm{d}V,$$

- Third order → four-modes coupling [Van Hoolst(1994)]

$$\kappa_{ABCD} = -\frac{1}{3!}\int_V \Big\{ \gamma(3-3\gamma+\gamma^2)p\,\eta_A\eta_B\eta_C\eta_D + 8\gamma p\,\eta_{[A}\eta_{BCD]} + 6\gamma(\gamma-2)p\,\eta_{[A}\eta_B\eta_{CD]} + \rho\,\xi_A^i\xi_B^j\xi_C^k\xi_D^l \nabla_i\nabla_j\nabla_k\nabla_l\Phi \Big\}\, \mathrm{d}V$$

- Fifth order → five-modes coupling

$$\kappa_{ABCDE} = \frac{1}{4!}\int_V \Big\{ \gamma(1+6\gamma-4\gamma^2+3\gamma^3)p\,\eta_A\eta_B\eta_C\eta_D\eta_E + 10\gamma^2(\gamma-3)p\,\eta_{[A}\eta_B\eta_C\eta_{DE]} + 15\gamma(\gamma-1)p\,\eta_{[A}\eta_{BC}\eta_{DE]} + 20\gamma^2 p\,\eta_{[A}\eta_B\eta_{CDE]} + 20\gamma p\,\eta_{[AB}\eta_{CDE]} - \rho\,\xi_A^i\xi_B^k\xi_C^l\xi_D^m\xi_E^n \nabla_i\nabla_k\nabla_l\nabla_m\nabla_n\Phi \Big\}\, \mathrm{d}V,$$

Slender tori

Example system: slender torus

- Polytropic equation of state:

$$\rho = \rho_0 f^n(r,z), \quad p = p_0 f^{n+1}(r,z)$$

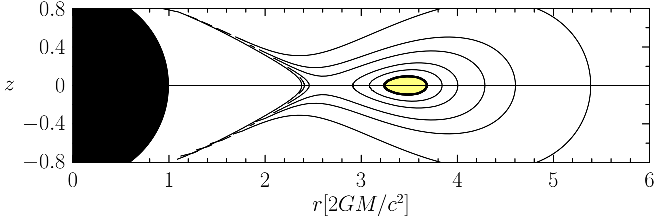
- Small filling parameter:

$$\beta^2 = 2(n+1)p_0/(\rho_0 r_0^2 \Omega_0^2) \sim (\Delta r/r_0)^2 \ll 1$$

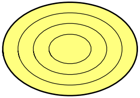
- the function f is expanded in the maximal pressure point:

$$\boxed{f = 1 - \left[\bar{\omega}_r^2 - \frac{2r_0}{\ell_0}\left(\frac{\mathrm{d}\ell}{\mathrm{d}r}\right)_0\right]\bar{x}^2 - \bar{\omega}_z^2 \bar{y}_0^2} \quad \text{...ellipses}$$

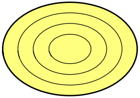
'shrinking' coordinates $\bar{x} \equiv (r - r_0)/\beta r_0$, $\bar{y} \equiv z/\beta r_0$



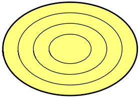
Lowest-order modes (Blaes et al, 2006)



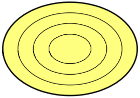
Radial epicyclic



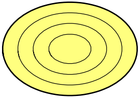
Vertical epicyclic



X-mode



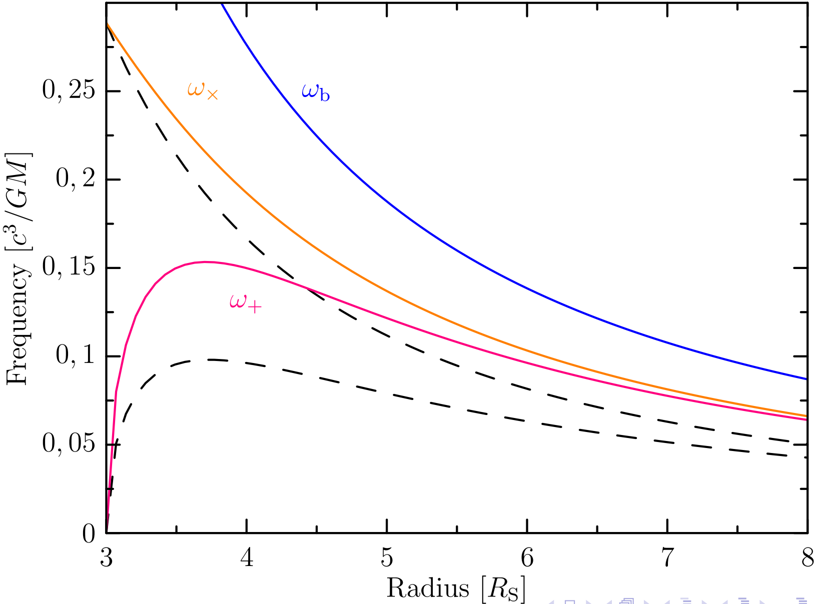
+-mode



Breathing mode

Eigenfrequencies...resonances

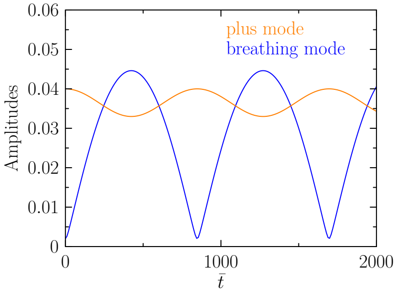
Polytropic index: $n = 3/2$

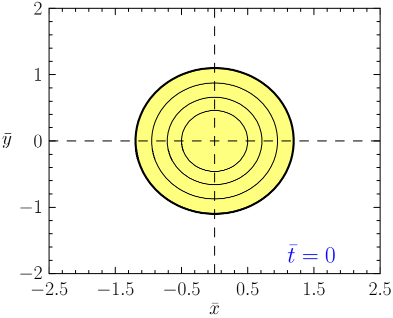


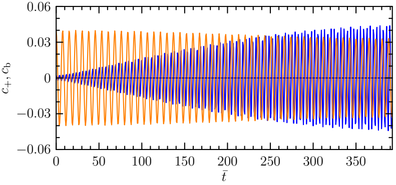
Example of the internal resonance

2:1 resonance

$$\frac{\omega_b}{\omega_+} = \frac{2}{1}$$







- Exchange of energy
- Low-frequency modulation

The 3:2 epicyclic resonance

Idea (Kluzniak & Abramowicz)

- Torus oscillating radially $\delta r(t) \propto \cos(\omega_r t)$
- Equation for vertical displacement:

$$\delta \ddot{z} + \omega_z^2[r_0 + \delta r(t)]\delta z = 0$$

- Mathieu equation

$$\delta \ddot{z} + \omega_z^2[1 + \epsilon \cos(\omega_z t)]\delta z = 0$$

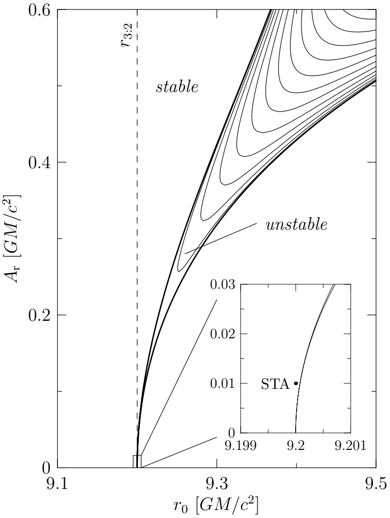
- Parametric resonance

$$\frac{\omega_z}{\omega_r} = \frac{n}{2}, \quad \text{first possible:} \quad n = 3$$

Resonance conditions

Order	General system	Epicyclic modes	
		resonance	resonance condition
2nd	1:2	1:2	$m_{\mathrm{r}} = 2m_{\mathrm{v}}$
	2:1		
3rd	1:3		
	1:1	1:1	$m_{\mathrm{r}} = m_{\mathrm{v}}$
	3:1		
4th	1:4	1:4	$m_{\mathrm{r}} = 4m_{\mathrm{v}}$
	2:3		
	3:2	**3:2**	$3m_{\mathrm{r}} = 2m_{\mathrm{v}}$
	4:1		

Resonance region



... step in γ*:* $0.25 \times 10^{-2}\Omega_0$

- Initial condition
  $\rightarrow$ torus at r_0
  $\rightarrow$ $A_{\rm r}$ = radial amplitude
- no feedback vert $\rightarrow$ rad
- Solution:

$$A_{\rm v} \propto {\rm e}^{\gamma t}$$

- Comparison to $t_{\rm th}$, $t_{\rm visc}$:

$$\boxed{\left(\frac{A_{\rm r}}{r_0}\right)^2 \gtrsim 10^{-6}\alpha_{\rm t}\beta^2}$$

- Turbulence...?

Resonance + turbulence

Stochastic excitation

$$\frac{\mathrm{d}c_A}{\mathrm{d}t} + \mathrm{i}\omega_A c_A = \frac{\mathrm{i}}{b_A}\left[\mathcal{F}_A(c_I) + Q_A(t)\right]$$

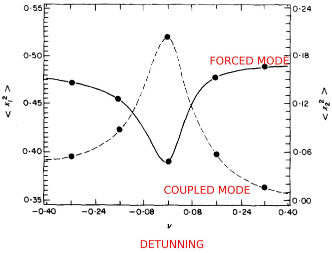
→ $Q_A(t)$ is a stochastic function

Inspiration:

Nayfeh & Serhan (1989):

→ 1:2 internal resonance

→ coupled oscillation excited



Three-mode resonances

and

nonlinear Papaloizou-pringle instability

Papaloizou-Pringle instability in slender tori

- Constant angular momentum distribution
- Expansion in the torus thickness [Blaes & Šrámková]:

$$\boxed{\beta \equiv \frac{\Delta r}{R}} \quad \omega = \omega^{(0)} + \beta\omega^{(1)} + \dots, \quad W = \frac{\delta p}{\rho\sigma} = W^{(0)} + \beta W^{(1)} + \dots.$$

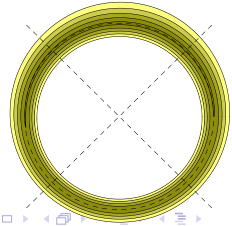
Corotation mode: Marginal stability

$$\omega_0 = \Omega_0 + \boxed{\mathrm{i}\sqrt{2}\, m\beta\, b}$$

$$W_0 = C_0 \left\{ 1 + m^2\beta^2 \left[a^2\bar{x}^2 - b^2\bar{y}^2 + \frac{4\sqrt{2}\,\mathrm{i}b}{\bar{\omega}_r^2}\bar{x} + \frac{\bar{\omega}_r^2 b^2 - \bar{\omega}_z^2 a^2}{2(n+1)\bar{\omega}_r^2\bar{\omega}_z^2} \right] + \mathcal{O}(\beta^3) \right\}$$

where

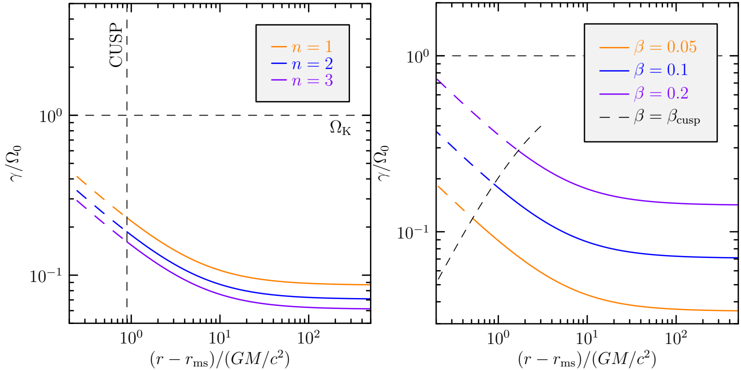
$$a^2 \equiv \frac{4(1+2n) + \bar{\omega}_r^2}{4(1+n)\bar{\omega}_r^2}, \quad b^2 \equiv \frac{4 - \bar{\omega}_r^2}{4(1+n)\bar{\omega}_r^2}.$$



$\Rightarrow$ Principal mode of the Papaloizou-Pringle

Growth-rates of the unstable mode

Dependence on the polytropic index and torus thickness



Nonlinear evolution: three-mode coupling

- Saturation by resonant interactions with damped modes
- Stars [Dziembowski 82, Moskalik 85, Nowakowski 05,...]

Common resonant triples ($m = 1$ corotation mode):

$$\begin{aligned} \delta\omega \equiv \omega_1 + \omega_2 - \omega_3 &\approx 0 = \omega + (\Omega_0 - \omega) - \Omega_0 + \mathcal{O}(\beta^2) \\ m_1 + m_2 - m_3 &= 0 = 0 + 1 - 1 \end{aligned}$$

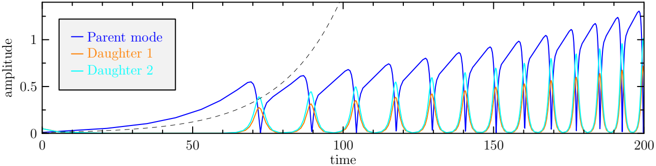
Amplitude equations

$$\begin{aligned} \dot{A}_1 &= \gamma_1 A_1 + \mathrm{i}\,\omega_1 \kappa A_2^* A_3 \mathrm{e}^{\mathrm{i}\delta\omega t} \\ \dot{A}_2 &= \gamma_2 A_2 + \mathrm{i}\,\omega_2 \kappa A_1^* A_3 \mathrm{e}^{\mathrm{i}\delta\omega t} \\ \dot{A}_3 &= \gamma_3 A_3 + \mathrm{i}\,\omega_3 \kappa A_1 A_2 \mathrm{e}^{-\mathrm{i}\delta\omega t} \end{aligned}$$

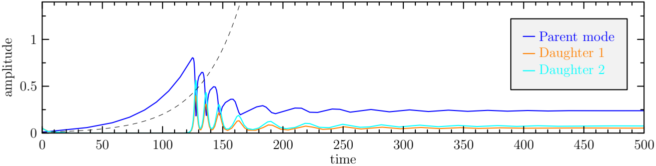
$\Rightarrow$ Unstable ‘parent’ mode ($\gamma_3 > 0$) $\rightarrow$ Damped ‘daughter’ modes ($\gamma_{1,2} < 0$).

Three-mode dynamics

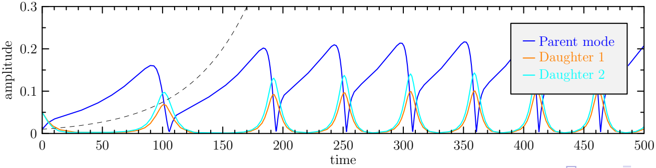
- $|\gamma_3| \gtrsim |\gamma_1| + |\gamma_2| \Rightarrow$ unstable



- $|\gamma_3| \ll |\gamma_1| + |\gamma_2|$ & $\delta\omega > \delta\omega_{\rm crit} \Rightarrow$ saturation



- $|\gamma_3| \ll |\gamma_1| + |\gamma_2|$ & $\delta\omega < \delta\omega_{\rm crit} \Rightarrow$ stable limit cycles



Observable consequence?

Three mode coupling condition: $\omega_1 + \omega_2 \approx \omega_3$

$\rightarrow$ XTE 1550-564:

$$92\text{Hz} + 184\text{Hz} = 276\text{Hz}$$

$\rightarrow$ GRS 1915+105: Fibonacci series (W.K.)

$$16\text{Hz} + 41\text{Hz} \approx 67\text{Hz}$$

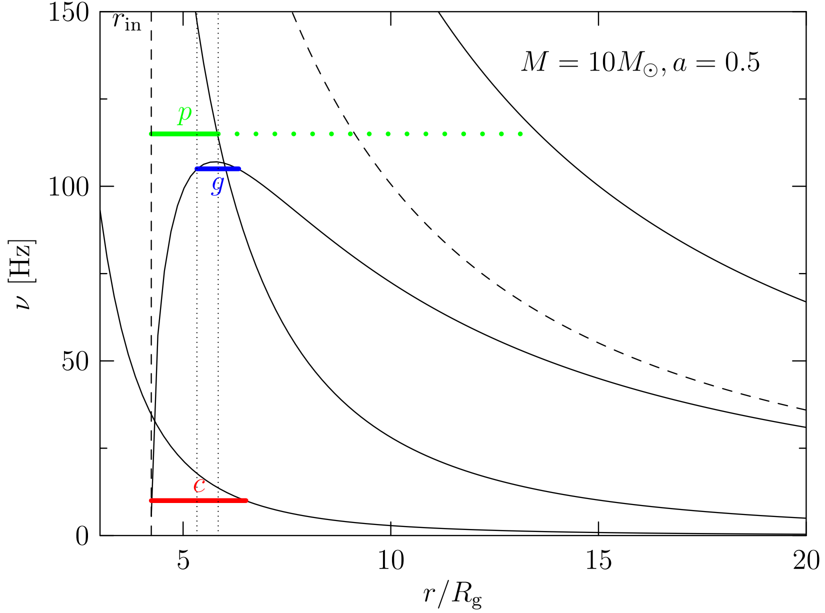
$$41\text{Hz} + 67\text{Hz} \approx 113\text{Hz}$$

$$\ldots$$

... 'Grand-daughter' modes (?)

Nonlinear diskoseismology

p-*g*-*c* coupling...?



Conclusions

Conclusions

- Unstable modes may be saturated by resonant processes
- Damped modes may reach substantial amplitudes
- Limit cycles: Low-frequency modulation (time scale $\propto 1/\gamma_3$)

Main question = damping

- α-viscosity seems to be insuficient
- MHD turbulence, disipation
- accretion
  (reduces excitation rate and increases damping rates)

Other quastions

- Role of the additional internal resonances (e.g. 1 : **2** : **3**)
- Saturation of other global instabilities.
  $\rightarrow$ MRI [linear analysis by Curry & Pudritz 95]