

Kluzniak-Kita expansion at the inner edge

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May 2009

Assumptions

- ▶ Stationary and axisymmetric disk: $\partial_t = \partial_\phi \equiv 0$
- ▶ Equatorial plane symmetry (odd/even functions of z)
- ▶ Polytropic equation of state: $P = K\rho^{1+1/n}$ with $n = 3/2$
- ▶ Height-dependent α -viscosity:

$$\nu = \frac{\alpha}{\Omega_K} \frac{c_s^2}{(1 + 1/n)}, \quad \eta = \nu\rho$$

- ▶ Geometrically thin disk: $|z| \ll r$
- ▶ Spherically symmetric gravitational potential $\Phi(\sqrt{r^2 + z^2})$:

$$\Phi(r, z) = \Phi(r, 0) + \frac{1}{2}\Omega_K^2(r)z^2 + \mathcal{O}(z^2/r^2), \quad |z| \ll r$$

Perturbative approach

Rescale all quantities by their typical values $Q \rightarrow \tilde{Q} \equiv Q/Q_*$:

- ▶ radius: $\tilde{r} = r/r_*$
- ▶ azimuthal velocities: $\tilde{\Omega} = \Omega/\Omega_K(r_*)$
- ▶ poloidal velocity: $\tilde{c}_s = c_s/c_{s*}$, $\tilde{v}^r = v^r/c_{s*}$, $\tilde{v}^z = v^z/c_{s*}$
- ▶ vertical coordinate: $\tilde{z} = z/H_*$
- ▶ density and pressure: $\tilde{\rho} = \rho/\rho_*$, $\tilde{p} = p/(c_{s*}^2 \rho_*)$
- ▶ viscosity: $\tilde{\nu} = \nu/(c_{s*} H_*)$, $\tilde{\eta} = \eta/(c_{s*} H_* \rho_*)$

Disk heigh: $H \sim c_s/\Omega_K \rightarrow$ Thin disk:

$$\epsilon = \frac{H_*}{r_*} = \frac{c_{s*}}{r_* \Omega_*} \ll 1$$

\rightarrow a small parameter. We assume $\tilde{Q} = \mathcal{O}(1)$ as $\epsilon \rightarrow 0$.

Hydrodynamical equations

radial:

$$\begin{aligned} & \epsilon^2 v^r \partial_r v^r + \epsilon v^z \partial_z v^r - r(\Omega^2 - \Omega_K^2) + \epsilon^2 n \partial_r c_s^2 + \frac{1}{2} \epsilon^2 \Omega_K' z^2 = \\ & = \epsilon^3 \frac{1}{r\rho} \partial_r (2r\eta \partial_r v^r) - 2\epsilon^3 \frac{\eta v^r}{\rho r^2} + \epsilon^2 \frac{1}{\rho} \partial_z (\eta \partial_z v^r) + \epsilon^2 \frac{1}{\rho} \partial_z (\eta \partial_r v^z) \\ & + \epsilon^3 \frac{1}{\rho} \partial_r \left[\left(\xi - \frac{2}{3} \eta \right) \frac{\partial_r (r v^r)}{r} \right] + \epsilon^2 \frac{1}{\rho} \partial_r \left[\left(\xi - \frac{2}{3} \eta \right) \partial_z v^z \right], \end{aligned} \quad (1)$$

azimuthal:

$$\epsilon \frac{v^r}{r^2} \partial_r (r^2 \Omega) + v^z \partial_z \Omega = \epsilon^2 \frac{1}{\rho} \partial_r (r^3 \eta \partial_r \Omega) + \frac{1}{\rho} \partial_z (\eta \partial_z \Omega), \quad (2)$$

vertical:

$$\begin{aligned} & \epsilon v^r \partial_r v^z + v^z \partial_z v^z + n \partial_z c_s^2 + \Omega_K^2 z = \\ & = \epsilon^2 \frac{1}{r\rho} \partial_r (r\eta \partial_r v^z) + \epsilon^2 \frac{1}{r\rho} \partial_r (r\eta \partial_z v^r) + \frac{1}{\rho} \partial_z (2\eta \partial_z v^z) \\ & + \epsilon^3 \frac{1}{\rho} \partial_z \left[\left(\xi - \frac{2}{3} \eta \right) \frac{\partial_r (r v^r)}{r} \right] + \epsilon^2 \frac{1}{\rho} \partial_z \left[\left(\xi - \frac{2}{3} \eta \right) \partial_z v^z \right], \end{aligned} \quad (3)$$

continuity equation:

$$\epsilon \frac{1}{r} \partial_r (r \rho v^r) + \partial_z (\rho v^z) = 0 \quad (4)$$

Successive solution

All quantities are expanded into series:

$$Q(r, z) = \sum_i \epsilon^i Q_i(r, z)$$

Equations are solved in each order of ϵ

Zeroth order results

- ▶ Radial equation \rightarrow angular velocity: $\Omega_0(r, z) = \Omega_K(r)$
- ▶ continuity equation \rightarrow vertical velocity: $v_0^z(r, z) = 0$
- ▶ vertical equation \rightarrow speed of sound:

$$c_{s0}^2(r, z) = \frac{1}{2n} \Omega_K^2 [H^2(r) - z^2] = \frac{1}{3} \Omega_K^2 [H^2(r) - z^2]$$

Other quantities (using standard expressions):

- ▶ density:

$$\rho_0 = \left(\frac{c_{s0}^2}{1 + 1/n} \right)^n = \left[\frac{\Omega_K^2}{5} (H^2 - z^2) \right]^{3/2}$$

- ▶ viscosity:

$$\eta_0 = \nu_0 \rho_0 = \frac{2}{3} \alpha \Omega_K^4 \left[\frac{1}{5} (H^2 - z^2) \right]^{5/2}$$

First order results

- ▶ Radial equation \rightarrow angular velocity: $\Omega_1(r, z) = 0$
- ▶ continuity equation \rightarrow vertical velocity: $v_1^z(r, z) = 0$
- ▶ azimuthal equation \rightarrow radial velocity: $v_0^r(r, z) = 0$
- ▶ vertical equation \rightarrow speed of sound: $c_{s0}^2(r, z) = 0$

Disk thickness

Angular momentum conservation:

$$\dot{M}(\ell - \ell_+) = -2\pi r^3 \int_{-\infty}^{\infty} \eta(\partial_r \Omega) dz,$$

takes the form in the lowest order of approximation

$$\dot{M}_1 [\ell_K(r) - \ell_+] = -4\pi r^3 \Omega'_K(r) \int_0^H \eta_0 dz$$

$$\dot{M}_1 = -4\pi r \int_0^H \rho_0 v_1^r dz.$$

From that we can derive the disk thickness as

$$H(r) = \left[\frac{3\lambda}{4 - \kappa} \frac{\ell_K - \ell_+}{r^2 \Omega_K^5} \right]^{1/6},$$

$$\kappa(r) \equiv 2 \frac{d \ln \ell_K}{d \ln r}, \quad \lambda \equiv 8 \cdot 5^{3/2} \pi^{-2} \left(\frac{\dot{M}_1}{\alpha} \right)$$

KK-solutions: $\kappa \equiv 1$; $\kappa \rightarrow 0$ as $r \rightarrow r_{\text{ms}}$

Second order results

- ▶ Coupled radial and azimuthal equations \rightarrow KK ansatz for v_1^r :

$$v_1^r(r, z) = f_1(r)(H^2 - z^2) + f_2(r)$$

- ▶ Solution for the radial velocity

$$v_1^r(r, z) = \alpha r \Omega_K \left(\frac{H}{r} \right)^2 \left[\Lambda \left(1 - \frac{z^2}{H^2} \right) + \frac{32\alpha^2 \Lambda}{15 \kappa} - \frac{2(4 - \kappa)}{3\kappa} \frac{d \ln H}{d \ln r} \right],$$

where

$$\Lambda(r) \equiv \frac{1}{15} \frac{(4 - \kappa)(16 - 5\kappa) + 2r\kappa'}{\kappa + 64\alpha^2/25}$$

KK-solutions: $\Lambda(r) \equiv \Lambda_{\text{KK}}$.

- ▶ accretion rate

$$\dot{M}_1 = \frac{\alpha \pi^2 \Omega_K^4 H^6}{12 \cdot 5^{3/2} \kappa} \left[r\kappa' + (4 - \kappa) \left(\frac{1}{2}(16 - 5\kappa) - 6 \frac{d \ln H}{d \ln r} \right) \right]$$

Second order results

- ▶ azimuthal velocity

$$\Omega_2(r, z) = \Omega_K \left(\frac{H}{r} \right)^2 \left[\frac{2}{15} \alpha^2 \Lambda \left(1 - 6 \frac{z^2}{H^2} \right) - \frac{4 - \kappa}{4} + \frac{1}{2} \frac{d \ln H}{d \ln r} \right].$$

- ▶ vertical velocity

$$v_2^z = \frac{1}{6} \alpha z \Omega_K \left(\frac{H}{r} \right)^2 \left\{ [2(4 - \kappa)\Lambda - r\Lambda'] \left(1 - \frac{z^2}{H^2} \right) + \left(\frac{\lambda}{\Omega_K^4 H^6} - 5\Lambda \right) \frac{d \ln H}{d \ln r} \right\},$$

Behavior of solutions close to the inner edge

Two singularities close to inner edge:

- ▶ $\kappa(r) \rightarrow 0$ as $r \rightarrow r_{\text{ms}}$
- ▶ $H(r) \rightarrow 0$ as $\ell_{\text{K}}(r) \rightarrow \ell_+ \approx \ell_{\text{ms}}$

Behavior of the solutions:

$$\begin{aligned}H(r) &\propto (\ell_{\text{K}} - \ell_+)^{1/6} \\ \rho(r) &\propto (\ell_{\text{K}} - \ell_+)^{1/2} \\ c_s^2(r) &\propto (\ell_{\text{K}} - \ell_+)^{1/3} \\ \eta(r) &\propto (\ell_{\text{K}} - \ell_+)^{5/6} \\ v^r(r) &\propto \epsilon (\ell_{\text{K}} - \ell_+)^{-2/3} \\ v^z(r) &\propto \frac{\epsilon^2 (r - r_{\text{ms}})}{(\ell_{\text{K}} - \ell_+)^{3/2}} \\ \Omega(r) - \Omega_{\text{K}}(r) &\propto \frac{\epsilon^2 (r - r_{\text{ms}})}{(\ell_{\text{K}} - \ell_+)^{2/3}}\end{aligned}$$

Solution (outline)

- ▶ l_+ is an 'eigenvalue' \Rightarrow Assumption: $l_+ \rightarrow l_{\text{ms}}$ as $\epsilon \rightarrow 0$

$$l_+ \equiv l_{\text{ms}} + \delta_+^2(\epsilon)L_+, \quad L_+ > 0$$

- ▶ disk solution breaks down when

$$r \rightarrow r_+ = r_{\text{ms}} + \delta_+ X_+, \quad X_+^2 = 2L_+(d^2\ell_K/dr^2)_{\text{ms}}$$

- ▶ Magnification: $r \equiv r_+ + \delta_r(\epsilon)X \Rightarrow \partial_r = \delta_r^{-1}\partial_X$.

- ▶ Rescaling:

$$H = (\delta_r\delta_+)\bar{H}, \quad \rho = (\delta_r\delta_+)^{1/2}\bar{\rho}, \quad c_s^2 = (\delta_r\delta_+)^{1/3}\bar{c}_s^2,$$

$$\eta = (\delta_r\delta_+)^{5/6}\bar{\eta}, \quad v^r = \epsilon(\delta_r\delta_+)^{-2/3}\bar{v}^r, \quad v^z = (\epsilon^2\delta_r^{-1/2}\delta_+^{3/2})\bar{v}^z$$

- ▶ Substitution into hdyn equations, balancing terms...
- ▶ Relation between ϵ , δ_r , δ_+ ??