

Oscillating magnetized Polish doughnuts

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Outline

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- ▶ Stationary magnetized tori in GR
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 - Slender-torus limit
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 - Modal equation
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 - Tori with constant β_p -parameter
 - Parker instability
- ▶ Discussion and conclusions

Stationary magnetized tori in general relativity

Rotating fluid with toroidal B -field

Relativistic Euler equation ($i = r, \theta$):

$$\mathcal{U}_{,i} - \frac{\Omega}{1 - \ell\Omega} \ell_{,i} + \frac{1}{w} p_{,i} + \frac{1}{2\mathcal{L}w} (\mathcal{L}b^2)_{,i} = 0$$

- ▶ $\mathcal{U} = \ln(u_t) = -\frac{1}{2} \ln(-g^{tt} + 2\ell g^{t\phi} - g^{\phi\phi})$ (eff. potential)
- ▶ $\ell = u_\phi / u_t$, (ang.momentum), $\Omega = u^\phi / u^t$ (ang. velocity)
- ▶ $w = e + p$ (specific enthalpy)
- ▶ $b^2 = b^\alpha b_\alpha$ (b-field in comoving frame), $b^\alpha u_\alpha = 0$
- ▶ $\mathcal{L} = g_{t\phi}^2 - g_{tt}g_{\phi\phi}$

Abramowicz, Jaroszynski & Sikora (1978), Komissarov (2006),
Montero, Zanotti, Font, Rezzolla (2007)

Integration – I

Relativistic Euler equation ($i = r, \theta$):

$$\mathcal{U}_{,i} - \frac{\Omega}{1 - \ell\Omega} \ell_{,i} + \frac{1}{w} \rho_{,i} + \frac{1}{2\mathcal{L}w} (\mathcal{L}b^2)_{,i} = 0$$

- ▶ Hydro (AJS78): Barotropic fluid

$$\rho = \rho(\rho) \quad \Rightarrow \quad \ell = \ell(\Omega)$$

- ▶ MHD (K06, MZFR07):

$$\rho = \rho(\rho) \quad \& \quad \ell = \ell(\Omega) \quad \& \quad (\mathcal{L}b^2) = (\mathcal{L}b^2)(\mathcal{L}w)$$

Polytropic relations: $\rho \propto \rho^{1+1/n}$, $\rho_m \propto \mathcal{L}^{\mu-1} w^\mu$

Integration – II

Integration (from the maximal-pressure point):

$$\mathcal{U} + \psi_\ell + \psi_g + \psi_m = \mathcal{U}_0$$

$$(\partial_r \mathcal{U})_{\ell,0} + (\partial_r \psi_m)_0 = 0$$

Potentials:

$$\psi_\ell = - \int_{\ell_0}^{\ell} \frac{\Omega d\ell}{1 - \ell\Omega}, \quad \psi_g = \int_{p_0}^p \frac{dp}{w}, \quad \psi_m = \int_{(\mathcal{L}b^2)_0}^{\mathcal{L}b^2} \frac{d(\mathcal{L}b^2)}{2\mathcal{L}w}$$

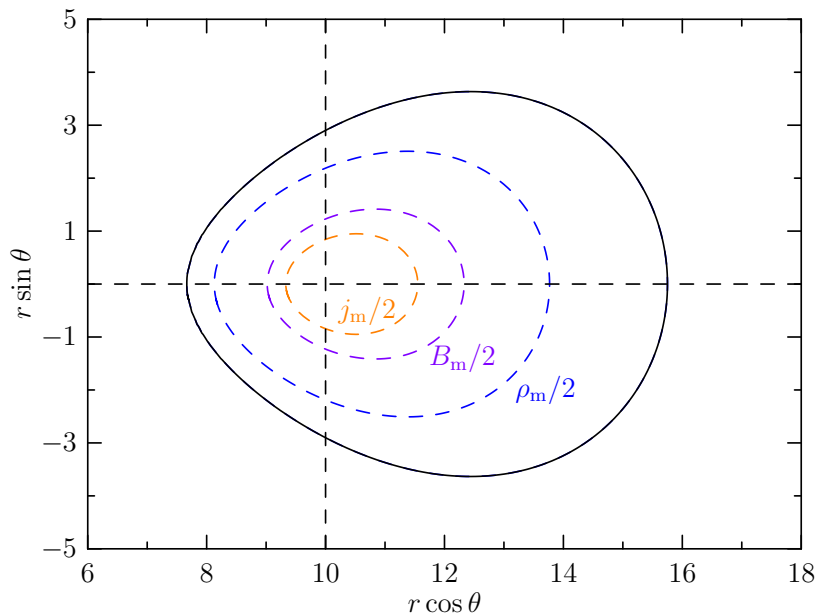
Polytropic relations: $\rho \propto \rho^{1+1/n}$, $\rho_m \propto \mathcal{L}^{\mu-1} w^\mu$ (MZFR07)

$$\rho = \rho_0 f^n, \quad p = p_0 f^{n+1} \quad b = b_0 \left(\frac{\mathcal{L}}{\mathcal{L}_0}\right)^{(\mu-1)/2} \left(\frac{1 + nc_{s0}^2 f}{1 + nc_{s0}^2}\right)^{\mu n/2}$$

Potentials:

$$\begin{aligned} \psi_\ell &= - \int_{\ell_0}^{\ell} \frac{\Omega d\ell}{1 - \ell\Omega}, \\ \psi_g &= \int_{p_0}^p \frac{dp}{w} = \ln \frac{1 + nc_{s0}^2 f}{1 + nc_{s0}^2}, \\ \psi_m &= \int_{(\mathcal{L}b^2)_0}^{\mathcal{L}b^2} \frac{d(\mathcal{L}b^2)}{2\mathcal{L}w} = \frac{\mu nc_{s0}^2}{(1 + nc_{s0}^2)\beta_p} \left[\left(\frac{1 + nc_{s0}^2 f}{1 + nc_{s0}^2}\right)^\mu \left(\frac{\mathcal{L}}{\mathcal{L}_0}\right)^{\mu-1} f^{(\mu-1)n} - 1 \right]. \end{aligned}$$

Distribution of various quantities



Slender-torus limit (Blaes 1985)

- ▶ Small central sound speed: $c_{s0} \ll r_0 \Omega_0 \sim c = 1$
- ▶ Expansion of ψ_g and ψ_m

$$\begin{aligned}\psi_g &= -nc_{s0}^2(1-f) + O(c_{s0}^4), \\ \psi_m &= \frac{\mu nc_{s0}^2}{\tilde{\beta}_p} [f^{(\mu-1)n} - 1] + O(c_{s0}^3).\end{aligned}$$

- ▶ Expansion of $\mathcal{U} + \psi_\ell$:

$$\mathcal{U} + \psi_\ell = \mathcal{U}_0 - (\partial_x \psi_m)_0 x + \frac{1}{2} r_0^2 (u_0^t)^2 [(\omega_{r0}^2 - \kappa_0^2) x^2 + \omega_{\theta 0}^2 y^2]$$

$$x = \frac{1}{r_0} g_{rr0}^{1/2} (r - r_0), \quad y = \frac{1}{r_0} g_{\theta\theta 0}^{1/2} (\pi/2 - \theta),$$

$$\omega_r^2 = \frac{(\partial_r^2 \mathcal{U})_{\ell=\text{const}}}{g_{rr}(u^t)^2}, \quad \omega_\theta^2 = \frac{(\partial_\theta^2 \mathcal{U})_{\ell=\text{const}}}{g_{\theta\theta}(u^t)^2}, \quad \kappa^2 = \left(\frac{u_t}{u^t}\right)^2 \frac{g_{,r}^{tt} - \ell g_{,r}^{t\phi}}{g_{rr}} (\ln \ell)_{,r}.$$

Slender tori

Equation describing torus structure

$$\frac{\tilde{\beta}_p f + \mu f^{(\mu-1)n}}{\tilde{\beta}_p + \mu} = \mathcal{F}(x, y) = 1 - \frac{1}{\beta^2} [(\bar{\omega}_{r0}^2 - \bar{\kappa}_0^2) x^2 + \bar{\omega}_{\theta 0}^2 y^2]$$

Slenderness parameter:

$$\beta^2 = \frac{2nc_{s0}^2(\mu + \tilde{\beta}_p)}{(u_0^t)^2 r_0^2 \Omega_0^2 \tilde{\beta}_p} \ll 1$$

Special cases:

- ▶ Hydro ($\tilde{\beta}_p \rightarrow \infty$) $\Rightarrow f = \mathcal{F}$
- ▶ Strong magnetization: ($\tilde{\beta}_p \rightarrow 0$) $\Rightarrow f = \mathcal{F}^{1/[n(\mu-1)]}$
- ▶ Constant β_p : ($\mu = 1 + 1/n$) $\Rightarrow f = \mathcal{F}$

Linear perturbations

Governing equations – I

Perturbations:

$$\delta q(x^\alpha) = \delta q(x^i) \exp[-i(\omega t - m\phi)], \quad \sigma(x^i) = \omega - m\Omega(x^i)$$

- ▶ Continuity equation \Rightarrow density

$$\boxed{\nabla_\alpha(\rho u^\alpha) = 0} \quad \Rightarrow \quad \delta\rho = -\frac{i}{\sigma u^t} \nabla_\alpha(\rho \delta u^\alpha)$$

- ▶ Induction equation \Rightarrow magnetic field

$$\boxed{\nabla_\alpha^* F^{\alpha\beta} = 0 = \nabla_\alpha(b^\alpha u^\beta - b^\beta u^\alpha)}$$

$$\Rightarrow \delta b^i = \frac{1}{\sigma u^t} (\ell\omega - m) b^\phi \delta u^i$$

$$\delta b^\phi = \frac{\omega b^\phi (1 - \ell\Omega)}{(\ell\omega - m) u_t} \delta u^\phi - \frac{i}{\ell\omega - m} \nabla_k \left(\frac{\ell\omega - m}{\sigma u^t} b^\phi \delta u^k \right).$$

$$\delta b^t = \ell \delta b^\phi - \frac{(1 - \ell\Omega) b_\phi}{u_t} \delta u^\phi.$$

Governing equations – II

- ▶ Equation of state $\delta p = c_s^2 \delta \rho \Rightarrow$ pressure, w , etc...

$$\Rightarrow \delta p = -\frac{ic_s^2}{\sigma u^t} \nabla_\alpha (\rho \delta u^\alpha), \quad \delta w = [1 + (1+n)c_s^2] \delta \rho$$

- ▶ Definition $a_\alpha \equiv u^\beta \nabla_\beta u_\alpha \Rightarrow$ four-acceleration

$$\Rightarrow \delta a_\alpha = u^\beta (\delta u_{\alpha,\beta} - \delta u_{\beta,\alpha}) + (u_{\alpha,\beta} - u_{\beta,\alpha}) \delta u^\beta$$

Governing equations – III

Euler equation

$$\nabla_{\beta} T_{\beta}^{\alpha} = 0$$

$$T_{\beta}^{\alpha} = (w + b^2)u^{\alpha}u_{\beta} + (\rho + \rho_m)\delta_{\beta}^{\alpha} - b^{\alpha}b_{\beta}$$

- ▶ (azimuthal component) + ℓ (time component) :

$$w \left[i\sigma u^t (1 - \ell\Omega) \delta u_{\phi} + u_t \ell_{,k} \delta u^k \right] + i(\omega\ell - m) \delta p + (b_{\phi,k} + \ell b_{t,k}) = 0$$

$\Rightarrow \delta u_{\phi}$ in terms of δu_i

- ▶ Poloidal components:

$$a_i \delta w + w \delta a_i + \delta p_{,i} + \delta \rho_{m,i} - b^{\alpha} \delta b_{i,\alpha} + g_{\alpha\beta,i} b^{\alpha} \delta b^{\beta} = 0$$

... substitution \Rightarrow single 2-d equation governing δu_i

Oscillations of magnetized slender tori

Modal equation

Eigen-value problem:

$$\boxed{(\bar{\sigma}_0^2 I + \hat{L}) \cdot \vec{U} = 0}$$

for the operator

$$\begin{aligned} \hat{L} \cdot \vec{U} = & -\bar{\kappa}_0^2 \vec{e}_{\bar{x}} \vec{e}_{\bar{x}} \cdot \vec{U} - \frac{1}{2} f^{-n} \vec{\nabla} \cdot (f^n \vec{U}) \vec{\nabla} \mathcal{F} + \\ & + \frac{f^{-n}}{\mu + \tilde{\beta}_p} \vec{\nabla} \cdot \left\{ (\mu - 1) f^{\mu n/2} \vec{\nabla} \cdot (f^{\mu n/2} \vec{U}) + \frac{\tilde{\beta}_p}{2n} f \vec{\nabla} \cdot (f^n \vec{U}) \right\} \end{aligned}$$

Poloidal velocity field $\vec{U} = (U^{\bar{x}}, U^{\bar{y}})$, $\bar{x} = x/\beta$, $\bar{y} = y/\beta$:

$$U^{\bar{x}} = \frac{1}{\beta r_0} g_{rr0}^{1/2} \delta u^r, \quad U^{\bar{y}} = -\frac{1}{\beta r_0} g_{\theta\theta 0}^{1/2} \delta u^\theta$$

\Rightarrow no difference between $m = 0$ and $m > 0$ modes

WKBJ analysis

Short wavelength perturbations: $|\vec{\nabla}\vec{U}| \gg |\vec{\nabla}f| \sim |\vec{\nabla}\mathcal{F}|$

$$\hat{L} \cdot \vec{U} = -\bar{\kappa}_0^2 \vec{e}_{\bar{x}} \vec{e}_{\bar{x}} \cdot \vec{U} + (\bar{c}_A^2 + \bar{c}_s^2) (\vec{\nabla} \cdot \vec{U}) \quad \bar{c}_{A,s} = \frac{c_{A,s}}{\beta u_0^t r_0 \Omega_0}$$

- ▶ Perturbation of the form

$$\vec{U} \propto \exp \left\{ i \int^{\bar{x}} k_{\bar{x}} d\bar{x} + i \int^{\bar{y}} k_{\bar{y}} d\bar{y} \right\},$$

- ▶ Dispersion relation

$$\bar{\sigma}_0^4 - [(\bar{c}_A^2 + \bar{c}_s^2) k^2 + \bar{\kappa}_0^2] \bar{\sigma}_0^2 + \bar{\kappa}_0^2 (\bar{c}_A^2 + \bar{c}_s^2) k_{\bar{y}}^2 = 0,$$

Solutions ($k \rightarrow \infty$):

$$\bar{\sigma}_0^2 = \bar{\kappa}_0^2 \left(\frac{k_{\bar{y}}}{k} \right)^2 \quad \text{and} \quad \bar{\sigma}_0^2 = (\bar{c}_A^2 + \bar{c}_s^2) k^2$$

... inertial and fast magnetosonic waves

Lowest-order modes – hydro (Blaes et al, 2006)

Radial epicyclic

Vertical epicyclic

X-mode

+ -mode

Breathing mode

Lowest-order modes – I

When \vec{U} is spatially constant:

$$\hat{L} \cdot \vec{U} = -(\bar{\omega}_{r0}^2 \vec{e}_x \vec{e}_x + \bar{\omega}_{\theta0}^2 \vec{e}_y \vec{e}_y) \cdot \vec{U}$$

→ two solutions – **epicyclic modes**

$$\bar{\sigma}_0^2 = \bar{\omega}_{r0}^2, \quad \vec{U} = (1, 0)$$

$$\bar{\sigma}_0^2 = \bar{\omega}_{\theta0}^2, \quad \vec{U} = (0, 1)$$

Similar trick when $\vec{\nabla} \cdot \vec{U} = 0$ (incompressible **X-mode**)

$$\bar{\sigma}_0^2 = \frac{1}{2} \left[\omega_r^2 + \omega_\theta^2 \pm \sqrt{(\omega_r^2 + \omega_\theta^2)^2 - 4k_0^2 \omega_\theta^2} \right] \quad \vec{U} = (y, x).$$

⇒ The same result as in hydro.

Special cases

PROBLEM: Complicated relation between f and \mathcal{F}

Special cases:

- ▶ Hydro ($\tilde{\beta}_p \rightarrow \infty$) $\Rightarrow f = \mathcal{F}$

$$\hat{L} \cdot \vec{U} = -\bar{\kappa}_0^2 \vec{e}_x \vec{e}_x \cdot \vec{U} + \frac{1}{2n} \vec{\nabla} [f^{-n+1} \vec{\nabla} \cdot (f^n \vec{U})]$$

- ▶ Strong magnetization: ($\tilde{\beta}_p \rightarrow 0$) $\Rightarrow f = \mathcal{F}^{1/[n(\mu-1)]}$

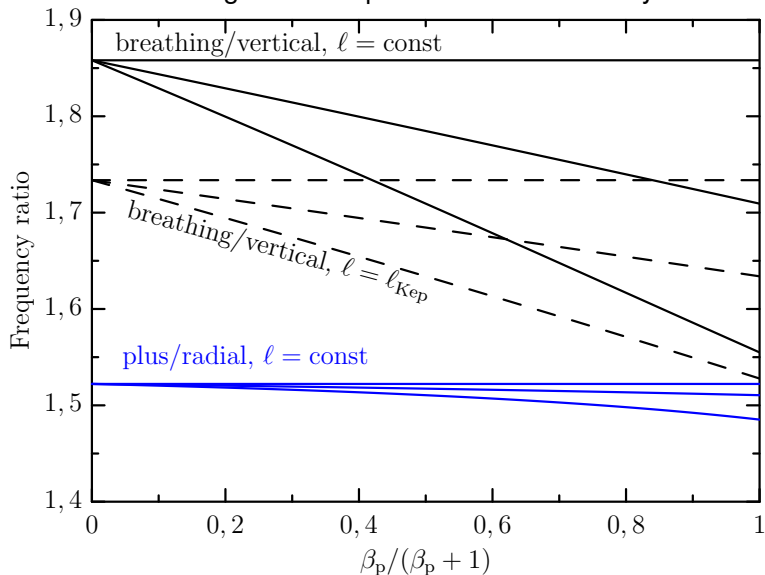
$$\hat{L} \cdot \vec{U} = -\bar{\kappa}_0^2 \vec{e}_x \vec{e}_x \cdot \vec{U} + \frac{1}{2} \vec{\nabla} [\vec{\nabla} \cdot (\mathcal{F} \vec{U})] + \frac{\mu-2}{2\mu} \mathcal{F} \vec{\nabla} (\vec{\nabla} \cdot \vec{U})$$

- ▶ Constant β_p : ($\mu = 1 + 1/n$) $\Rightarrow f = \mathcal{F}$

$$\hat{L} \cdot \vec{U} = -\bar{\kappa}_0^2 \vec{e}_x \vec{e}_x \cdot \vec{U} + \frac{1}{2n} \vec{\nabla} [f^{1-n} \vec{\nabla} \cdot (f^n \vec{U})] + \frac{n-1}{2n[1+n(\tilde{\beta}_p+1)]} f^{-n} \vec{\nabla} (f^{n+1} \vec{\nabla} \cdot \vec{U})$$

Lowest-order modes – II

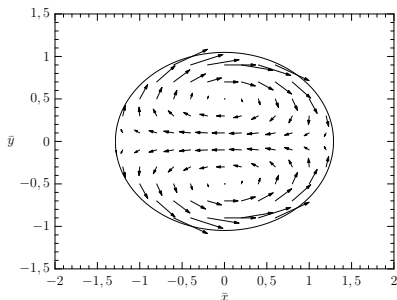
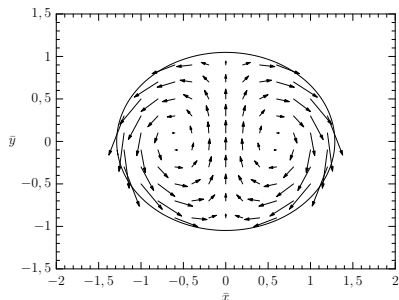
Plus and breathing mode frequencies are affected by b -field.



... constant β_p torus.

Parker instability – I

Second-order modes of strongly magnetized tori → INSTABILITY



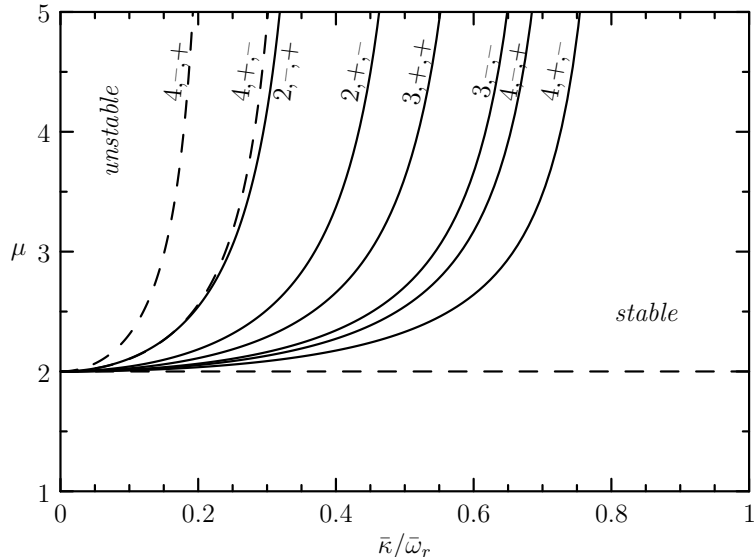
Instability criteria: ($k = \bar{k}_0 / \bar{\omega}_{r0}$)

- ▶ (+, -)-mode: $\mu = \frac{6-8k^2}{3-10k^2}$
- ▶ (-, +)-mode: $\mu = \frac{6-14k^2+8k^4}{3-22k^2+16k^4}$.

No rotation: $-\partial_{x,y} \ln(B/\rho) > 0 \Leftrightarrow \mu > 2$

Parker instability – II

Higher-order modes are stabilized (Coriolis force?)



⇒ With increasing wavenumber the torus becomes more unstable

Concussions

Conclusions

- ▶ Simple probe to the complicated dynamics of accretion disks
→ Parker instability
- ▶ Other instabilities (PP, MRI): suppressed
→ perturbative expansion in β (Blaes 1985, Blaes+2006)