Internal resonance and the amplitude evolution of neutron-star QPOs

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Abstract

We study the properties of twin kilohertz quasiperiodic oscillations (QPOs) with a simple toy model consisting of two oscillation modes coupled with a general nonlinear force. We examine resonant effects by slowly varying the values of the tunable, and nearly commensurable, eigenfrequencies. The behavior of the true oscillation frequencies and amplitudes during a slow transition through the 3:2 resonance is examined in detail, and it is shown that both are affected significantly by the nonlinearities in the governing equations. In particular, the amplitudes of oscillations reflect a resonant exchange of energy between the modes, and as a result the initially weaker mode may become dominant after the transition. We note that a qualitatively similar behavior was reported for several neutron-star sources where the difference in the amplitudes of neutron-star twin-peak QPOs changes sign as the observed frequency ratio of the QPOs passes through the value 3:2.

References

- Horák J., Abramowicz M. A., Kluźniak W., Rebusco P. & Török G. 2009, A&A **499**, 535
- Török G. 2009, A&A **497**, 661

Evolution of QPO-amplitudes in neutron stars

In neutron-star sources, kHz QPOs often arise as two simultaneously observed peaks with frequencies that change over time. The twin kHz QPOs usually span a wide frequency range and follow a nearly linear $\nu_u - \nu_\ell$ relation for each source. Recently, Török (2009) studied the difference in strength between the twin kHz QPOs, $r = r_\ell - r_u$, defined as the difference between the fractional rms amplitudes of the lower and upper kHz QPOs.

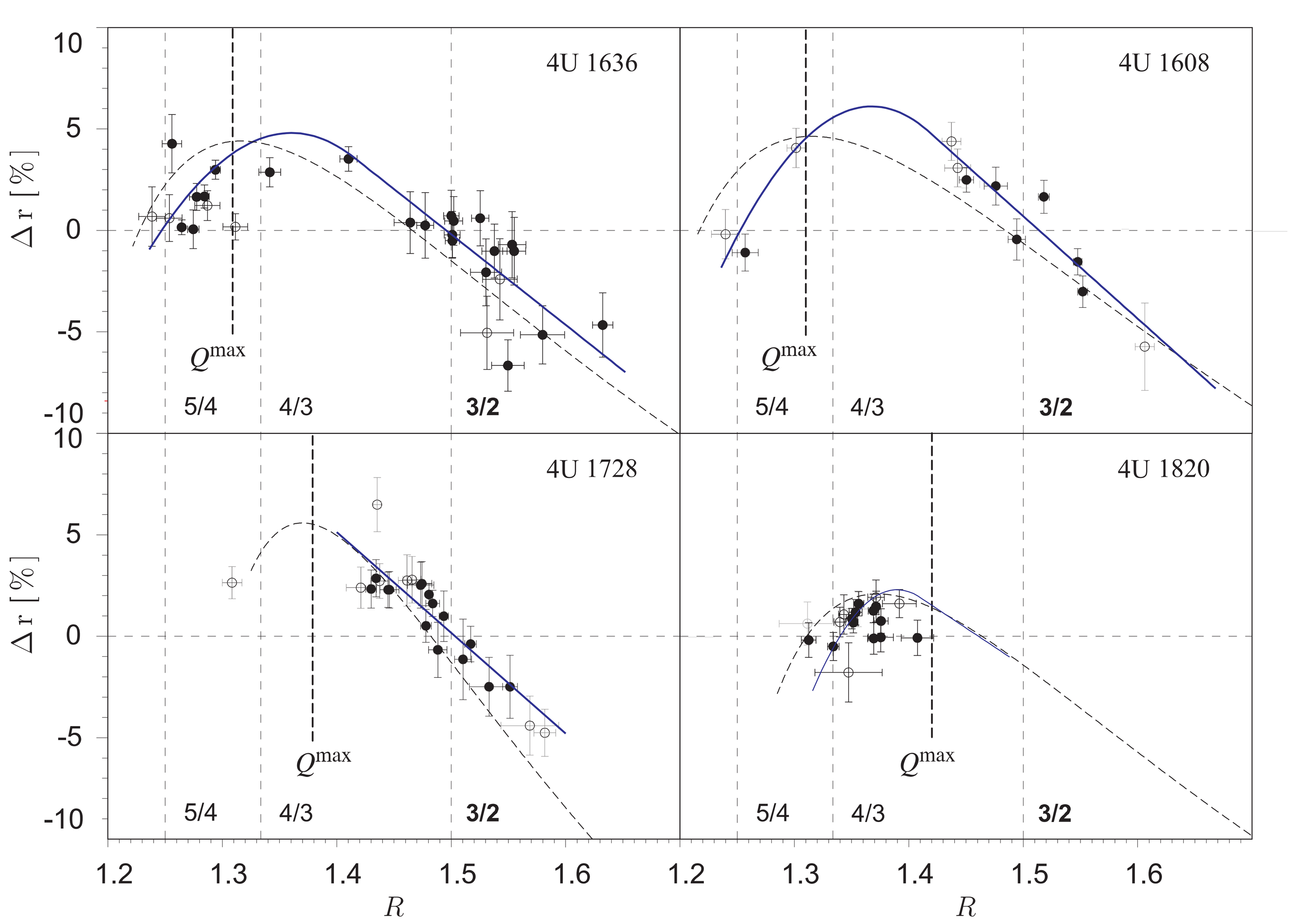

Figure 1: The dependence of the amplitude difference on the frequency ratio in the four atoll sources, with detailed view shown of the range $\mathcal{R}^\star = 1.2 - 1.7$. The open points are above the 2.5σ threshold; the filled points have significance greater than 3σ. The vertical lines labelled with $Q^{\max}$ denote the frequency ratio corresponding to the maximum of the lower QPO quality factor.

Toy model

We model the QPOs using two coupled, nonlinear oscillators governed by the equations

$$\ddot{x}_\ell + \omega_\ell(t)^2 x_\ell = f_\ell(x_\ell, x_u, \dot{x}_\ell, \dot{x}_u), \quad (1)$$
$$\ddot{x}_u + \omega_u(t)^2 x_u = f_u(x_\ell, x_u, \dot{x}_\ell, \dot{x}_u). \quad (2)$$

The upper and lower QPOs are identified with the solutions $x_u(t)$ and $x_\ell(t)$, respectively. The coupling functions f_ℓ and f_u are nonlinear and invariant under time inversion. The slowly varying functions $\omega_\ell > 0$, and $\omega_u > 0$ are the eigenfrequencies of the two oscillators and we further concentrate on the case when ratio $\mathcal{R} \equiv \omega_u/\omega_\ell$ is close to 3:2. The degree of closeness to the rational ratio is expressed by the detuning parameter $\sigma \equiv 2\omega_u - 3\omega_\ell$.

Dynamics

When the amplitudes of oscillations and the detuning parameter are small, the lowest-order real displacements are given by

$$x_\ell(t) = \mathrm{Re}\left[A_\ell(t) e^{i\omega_\ell t}\right], \quad x_u(t) = \mathrm{Re}\left[A_u(t) e^{i\omega_u t}\right], \quad (3)$$

where the complex amplitudes

$$A_\ell(t) \equiv \frac{1}{2} a_\ell(t) e^{i\phi_\ell(t)}, \quad A_u(t) \equiv \frac{1}{2} a_u(t) e^{i\phi_u(t)} \quad (4)$$

depend slowly on time. The true ('observed') frequencies of the solution are therefore shifted with respect to the eigenfrequencies by the corrections $\Delta\omega_\ell = \dot{\phi}_\ell$ and $\Delta\omega_u = \dot{\phi}_u$. The dynamics of the system is governed by the two first-order ODEs

$$\dot{\xi} = \frac{1}{8}\beta\omega_u(1-\xi)(\xi E)^{3/2}\sin\gamma, \quad (5)$$
$$\dot{\gamma} = \sigma + \frac{1}{4}\omega_u E\left[\mu_\ell \xi + \frac{\mu_u}{\nu}(1-\xi) + \frac{1}{4}\beta(3-5\xi)(\xi E)^{1/2}\cos\gamma\right], \quad (6)$$

in which $\xi \equiv a_\ell/E$, $E \equiv a_\ell^2 + \nu a_u^2$ is conserved energy, $\gamma \equiv 2\phi_u - 3\phi_\ell + \sigma t$, and β, μ_u, and μ_ℓ are as yet unspecified dimensionless constants characterizing a particular system.

Phase-plane topologies for the 3:2 resonance

For fixed eigenfrequencies, depending on the energy of oscillations E and eigen-frequency ratio $\mathcal{R}$, there may exist **zero, one or two** fixed points in this disk, each representing strictly periodic oscillations of the system. They correspond to oscillations whose amplitudes are not modulated and whose phases $\phi_\ell(t)$ and $\phi_u(t)$ are linear functions of time that adjust the frequencies of oscillations to the exact 3:2 ratio.

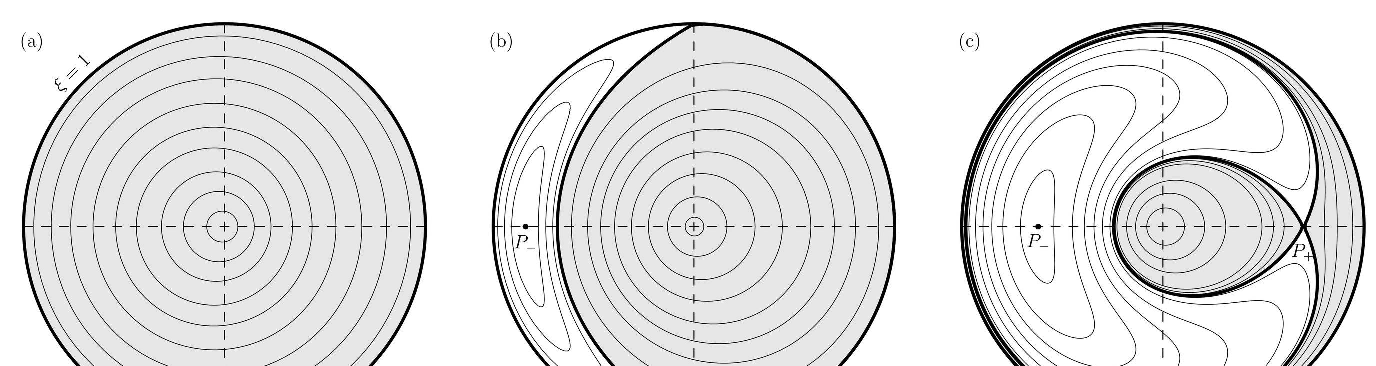

Figure 2: The three possible types of topologies of the (γ, ξ)-disk. The polar angle and the distance from the origin represent the phase function γ and the fractional energy ξ, respectively. While the panel (a) shows the phase-plane out of the resonance, the cases (b) and (c) correspond to the resonance. In these cases the phase-planes contain also the librating trajectories (white region).

Slow passage through the resonance

We consider the case when the eigenfrequencies vary slowly and the system slowly gradually passes through the resonance keeping the total energy of oscillations constant. One example is shown in the Fig. 3 and 4. During the transition, the topology of the phase plane gradually changes in a sequence (A)→(C)→(B)→(A).

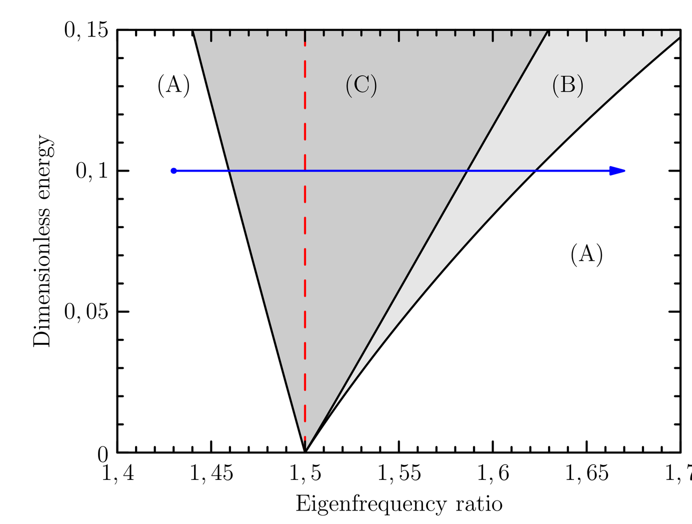

Figure 3: Regions where a resonance is present (resonance 'tongues') in the plane of the eigenfrequency ratio and the total energy of oscillations. Each point in the plane corresponds to one single (γ, ξ)-disk and different regions to different topologies of these disks.

Outside the resonance the phase plane initially contains only circulating orbits. The ξ-coordinate oscillates slightly in correspondence to a weak energy exchange between the modes. When the system enters into the resonance tongue, the energy exchange becomes more prominent and asymmetric i.e., the inflow of the energy into one of the modes exceeds the outflow. After the transition the energy exchange between modes becomes more symmetric and less apparent. The final fractional energy ξ_f after the passage is generally different from the initial one ξ_i: the energy is redistributed due to the resonance.

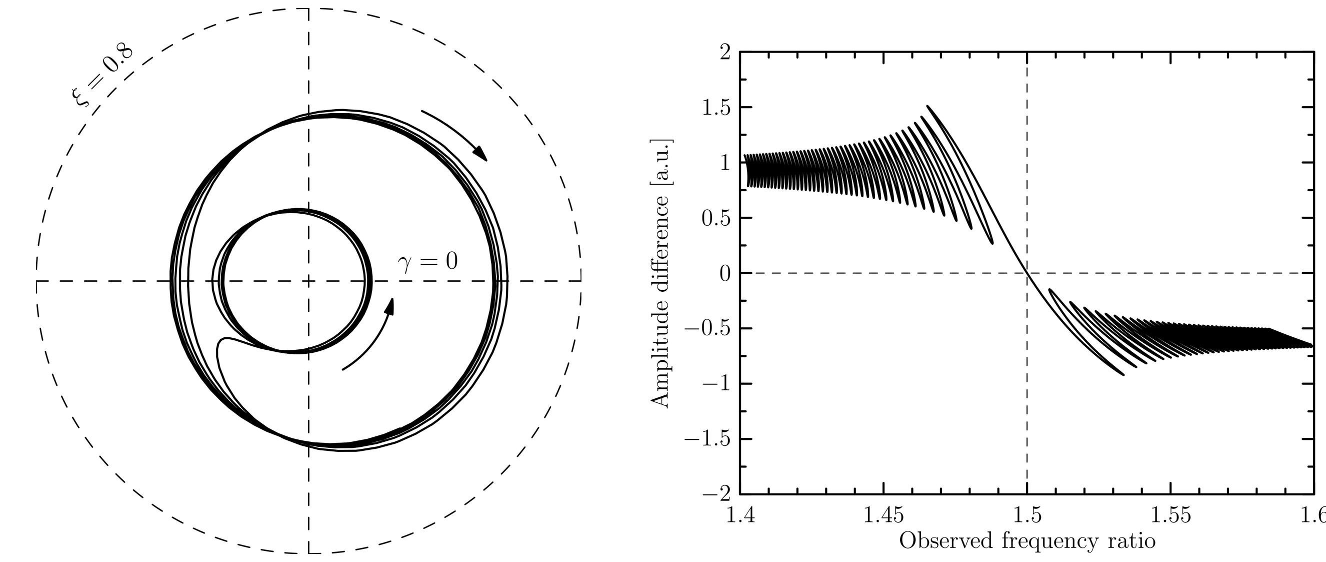

Figure 4: Transition of the resonance. The left panel shows the trajectory of the system projected into the (γ, ξ)-disk. The right panel shows behavior of the amplitude difference, $\delta a = a_\ell - a_u$, (measured in arbitrary units) as a function of the ratio the observed frequencies. Because of the slow changes in the eigenfrequencies, the energy exchange between modes is not exactly symmetric and the total energy is redistributed between the modes when the system passed the resonance.

Conclusion

- Assuming a slow (secular) change of the eigenfrequencies of the two coupled oscillators, we demonstrated that the energy is redistributed between the two oscillators when the system passes through a resonance.
- This effect is evident in the change in the sign of the amplitude difference when the ratio of the eigenfrequencies ratio crosses a rational value, such as 3:2.
- This result is in qualitative agreement with the observations for some neutron stars (compare Fig 1 and 4).