

# Evolution of viscous slender tori

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14–16.9.2010, Opava

# Outline

- ▶ Inviscid slender tori
  - Construction of polytropic accretion tori
  - Slender-torus limit
- ▶ Secular evolution of viscous tori
  - Viscosity prescription
  - Equation governing a slow evolution and their solutions
- ▶ Perturbations
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- ▶ Results for the lowest order modes
  - Effects of viscosity prescription
  - Various angular-momentum distributions
  - Dynamics of the inertial mode
- ▶ Application: kHz QPOs in neutron stars
  - $\nu$ -Q relations for lowest order modes

# Inviscid accretion tori

## Accretion tori

Rotating fluid ( $v^i = \Omega \delta_\phi^i$ ,  $\Omega = \ell/r^2$ ):

$$\nabla^i \Phi - \frac{\ell^2}{r^3} \delta_r^i = \frac{1}{\rho} \nabla^i p,$$

Polytrope  $p = p(\rho) \propto \rho^{1+1/n} \Rightarrow$  RHS is a gradient,

$$RHS = \frac{1}{\rho} \nabla^i p = n \nabla^i c_s^2 = n c_{s0}^2 \nabla^i f, \quad (0 \leq f \leq 1)$$

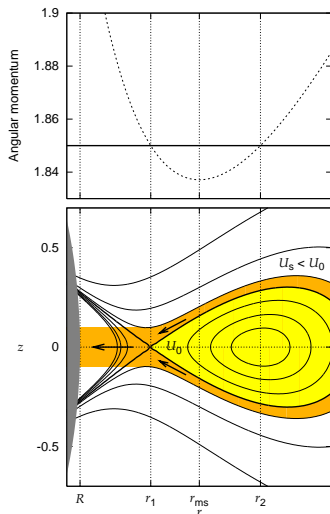
and therefore LHS is a gradient ( $\Rightarrow \ell = \ell(r)$  only),

$$LHS = \nabla^i \Phi - \frac{\ell^2}{r^3} \delta_r^i = \nabla^i \left( \Phi + \frac{\ell_0^2}{2r^2} \right) - \frac{\ell^2 - \ell_0^2}{r^3} = \nabla^i \mathcal{U} + \nabla^i \Psi.$$

This can be integrated to

$$f = 1 - \frac{1}{n c_{s0}^2} (\mathcal{U} - \mathcal{U}_0 + \Psi)$$

# Torus structure



Pressure and density are given by  $f$ ,

$$f = 1 - \frac{1}{nc_{s0}^2} (\mathcal{U} - \mathcal{U}_0 + \Psi).$$

- ▶  $f = 0 \Rightarrow$  surface of the torus
- ▶  $f = 1 \Rightarrow$  torus center

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- ▶  $\ell(r) \rightarrow \mathcal{U}, \Psi \Rightarrow$  torus shape
  - ▶  $c_{s0} \Rightarrow$  torus size

$$\Delta r, \Delta z \sim c_{s0}/\Omega_0$$

## Slender-torus limit

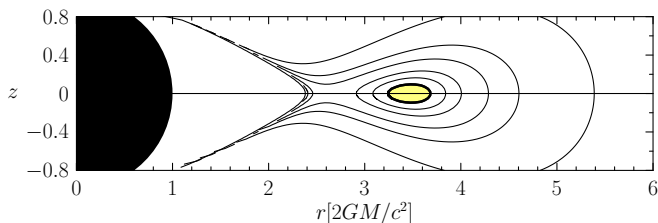
Slenderness parameter and local coordinates

$$\beta = \sqrt{2n} \frac{c_{s0}}{r_0 \Omega_0} \sim \frac{\Delta r}{r_0} \sim \frac{\Delta z}{r_0}, \quad \bar{x} = \frac{r - r_0}{\beta r_0} \quad \bar{y} = \frac{z}{\beta r_0}$$

when  $\beta \rightarrow 0$

$$f \approx 1 - \frac{\beta^2 r_0^2}{2nc_{s0}^2} \left[ \left( \mathcal{U}_{rr} - \frac{d \ln \ell^2}{d \ln r} \right)_0 \bar{x}^2 + (\mathcal{U}_{zz})_0 \bar{y}^2 \right] = 1 - (\bar{\omega}_r^2 - \bar{\kappa}^2) \bar{x}^2 - \bar{\omega}_z^2 \bar{y}^2$$

$\bar{\omega}_{r,z}$  = radial/vertical epicyclic frequency normalized by  $\Omega_0$ .



# Viscous tori

# Viscosity prescription

Navier-Stokes equation

$$\frac{\partial v^i}{\partial t} + v^k \nabla_k v^i + \frac{1}{\rho} \nabla^i \rho + \nabla^i \Phi = \frac{1}{\rho} \nabla_k \sigma^{ki} = \mathcal{F}_{\text{visc}}^i$$

$\sigma^{ik}$  is the viscous stress tensor (Landau & Lifshitz):

$$\sigma^{ik} = \eta (\nabla^k v^i + \nabla^i v^k) + \left( \xi - \frac{2}{3} \eta \right) (\nabla \cdot \mathbf{v}) g^{ik}$$

►  $\eta$  **dynamic viscosity** coefficient,  $\xi$  **bulk viscosity** coefficient

' $\alpha p$ '-parameterization (Shakura & Sunyaev):

$$\eta = \frac{\alpha p}{\Omega_0}, \quad \left( \xi - \frac{2}{3} \eta \right) = a \frac{\alpha p}{\Omega_0}, \quad \alpha \ll 1$$

$\alpha$  is a small (perturbation) parameter in our analysis

# Secular evolution – I

We assume a slow poloidal flow

$$v^i \sim \alpha r \Omega \quad (i = r, z)$$

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- ▶ Poloidal components ( $O(1)$ )  $\Rightarrow$  **Torus structure**

$$\frac{1}{\rho} \nabla^i p + \nabla^i \Phi - r \Omega^2 \delta_r^i = 0,$$

- ▶ Azimuthal component ( $O(\alpha)$ )  $\Rightarrow$  **Evolution of ang. momentum**

$$\frac{\partial \ell}{\partial t} = -v^r \frac{\partial \ell}{\partial r} + \frac{\alpha}{r \rho} \frac{\partial}{\partial r} \left( r^2 \rho \frac{d \ln \Omega}{d \ln r} \right)$$

- ▶ Continuity equation ( $O(\alpha)$ )  $\Rightarrow$  **Evolution of the density**

$$\frac{\partial \rho}{\partial t} = -\frac{\partial}{\partial r} (\rho v^r) - \frac{\partial}{\partial z} (\rho v^z)$$

## Secular evolution – II

Same structure as in the inviscid case

$$\rho = \rho_0 f^n, \quad p = p_0 f^{n+1}, \quad f = 1 - (\bar{\omega}_r^2 - \bar{\kappa}^2) \bar{x}^2 - \bar{\omega}_z^2 \bar{y}^2,$$

but  $\rho_0$ ,  $p_0$ ,  $\beta$  and  $\bar{\kappa}$  are **slow functions of time**.

Their evolution follows from the two other equations

$$\frac{d \ln \beta^2}{dt} + \frac{d \bar{\kappa}^2}{dt} \bar{x}^2 + \frac{f^{n-1}}{n \beta r_0} \left[ \frac{\partial}{\partial \bar{x}} (f^n v^r) + \frac{\partial}{\partial \bar{y}} (f^n v^z) \right] = 0$$

and

$$\left[ \frac{d \bar{\kappa}^2}{dt} - \alpha \frac{\ell_0}{r_0^2} (4 - \bar{\kappa}^2) (\bar{\omega}_r^2 - \bar{\kappa}^2) \right] \beta \bar{x} + \frac{1}{r_0} \bar{\kappa}^2 v^r = 0.$$

## Secular evolution – III

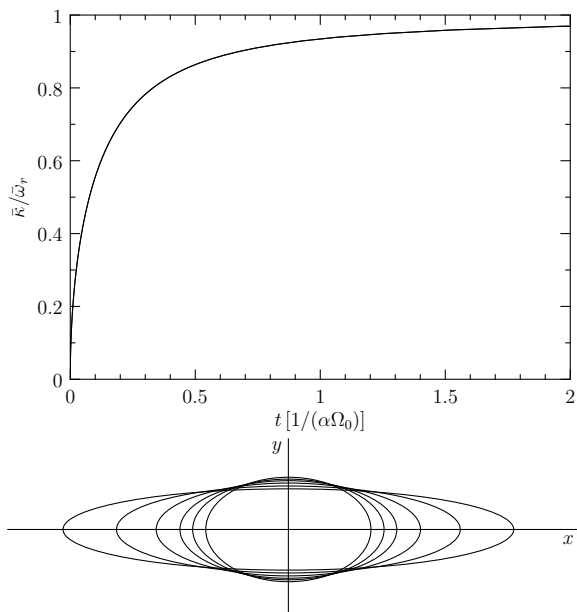
Ansatz for the solution

$$v^r(t, \bar{x}, \bar{y}) = \beta \bar{x} V^r(t), \quad v^z(t, \bar{x}, \bar{y}) = \beta \bar{y} V^z(t),$$

then one obtains a close system of equations

$$\begin{aligned} n \frac{d \ln \beta^2}{dt} &= -\frac{1}{r_0} (V^r + V^z), \\ n \frac{d \bar{k}^2}{dt} &= \frac{1}{r_0} [(2n + 1) V^r + V^z] (\bar{\omega}_r^2 - \bar{k}^2), \\ 0 &= \frac{1}{r_0} [V^r + (2n + 1) V^z] \bar{\omega}_z^2, \\ \frac{d \bar{k}^2}{dt} &= \frac{\alpha \ell_0}{r_0^2} (4 - \bar{k}^2) (\bar{\omega}_r^2 - \bar{k}^2) - \frac{1}{r_0} \bar{k}^2 V^r \end{aligned}$$

## Secular evolution – solution



# Perturbations

# Master equation

Master equation for the oscillation modes

$$\bar{\sigma}^4 + \bar{\sigma}^2 \hat{B}W + \hat{C}W = i\alpha \hat{F}(\bar{\sigma})W$$

- ▶ Eigenfunctions: Papaloizou & Pringle variable  $W = -\delta p / (\rho \sigma)$
- ▶ Eigenfrequencies: Corotation frequency  $\sigma = \omega - m\Omega_0$
- ▶  $\hat{A}$ ,  $\hat{B}$ : linear differential operators

$$\hat{B} = \frac{f^{1-n}}{2n} \left[ \frac{\partial}{\partial \bar{x}} \left( f^n \frac{\partial}{\partial \bar{x}} \right) + \frac{\partial}{\partial \bar{y}} \left( f^n \frac{\partial}{\partial \bar{y}} \right) \right] - \bar{k}$$
$$\hat{C} = -\bar{k}^2 \frac{f^{1-n}}{2n} \frac{\partial}{\partial \bar{y}} \left( f^n \frac{\partial}{\partial \bar{y}} \right)$$

- ▶  $\alpha \hat{F}$ : a small perturbation of the eigenvalue problem
- ▶  $\alpha \hat{F} = 0$ : inviscid-torus problem

## Master equation

$$\begin{aligned}\hat{F}(\bar{\sigma}) = & -\frac{\bar{\sigma}^2 - \bar{k}^2}{4n(n+1)\bar{\sigma}f^{n-1}} \left\{ a \left[ \frac{\partial^2}{\partial \bar{x}^2} \left( f^{n+1} \frac{\partial^2}{\partial \bar{y}^2} \right) + \frac{\partial^2}{\partial \bar{y}^2} \left( f^{n+1} \frac{\partial^2}{\partial \bar{x}^2} \right) \right] \right. \\ & + (2+a) \left[ \frac{\partial^2}{\partial \bar{x}^2} \left( f^{n+1} \frac{\partial^2}{\partial \bar{x}^2} \right) \right. \\ & \left. \left. + \frac{\partial^2}{\partial \bar{y}^2} \left( f^{n+1} \frac{\partial^2}{\partial \bar{y}^2} \right) \right] + 4 \frac{\partial^2}{\partial \bar{x} \partial \bar{y}} \left( f^{n+1} \frac{\partial^2}{\partial \bar{x} \partial \bar{y}} \right) \right. \\ & - 2(n+1)(4 - \bar{k}^2) \frac{\partial}{\partial \bar{x}} \left( f^{n+1} \frac{\partial}{\partial \bar{x}} \right) + \\ & \frac{\bar{k}^2}{\bar{\sigma}^2 - \bar{k}^2} \left[ (3+a) \frac{\partial^2}{\partial \bar{x}^2} \left( f^{n+1} \frac{\partial^2}{\partial \bar{x}^2} \right) \right. \\ & \left. + 3 \frac{\partial^2}{\partial \bar{x} \partial \bar{y}} \left( f^{n+1} \frac{\partial^2}{\partial \bar{x} \partial \bar{y}} \right) \right. \\ & \left. \left. + a \frac{\partial^2}{\partial \bar{y}^2} \left( f^{n+1} \frac{\partial^2}{\partial \bar{x}^2} \right) \right] \right\}.\end{aligned}$$

# Perturbation approach to perturbations

A small change of the eigenfunctions and eigenfrequencies due to the viscosity

$$\begin{aligned}\bar{\sigma}_\nu &= \bar{\sigma}_\nu^{(0)} + \alpha \bar{\sigma}_\nu^{(1)} + \dots, \\ W_\nu &= W_\nu^{(0)} + \alpha \sum_\mu c_{\nu\mu} W_\mu^{(0)} + \dots\end{aligned}$$

First-order correction

$$\bar{\sigma}_\nu^{(1)} = \frac{i}{2\bar{\sigma}_\nu^{(0)}} \frac{\langle W_\nu^{(0)}, \hat{F} W_\nu^{(0)} \rangle}{\langle W_\nu^{(0)}, (\hat{B} + 2\sigma_\nu^{(0)2}) W_\nu^{(0)} \rangle}$$

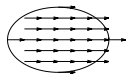
with the scalar product defined as

$$\langle U, V \rangle = \int U^* V f^n d\bar{x} d\bar{y}.$$

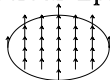
## Damping of the lowest-order modes

# Lowest-order modes (Blaes, 2006)

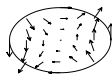
Radial Epicyclic



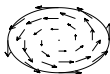
Vertical Epicyclic



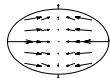
X Mode



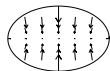
Inertial Mode



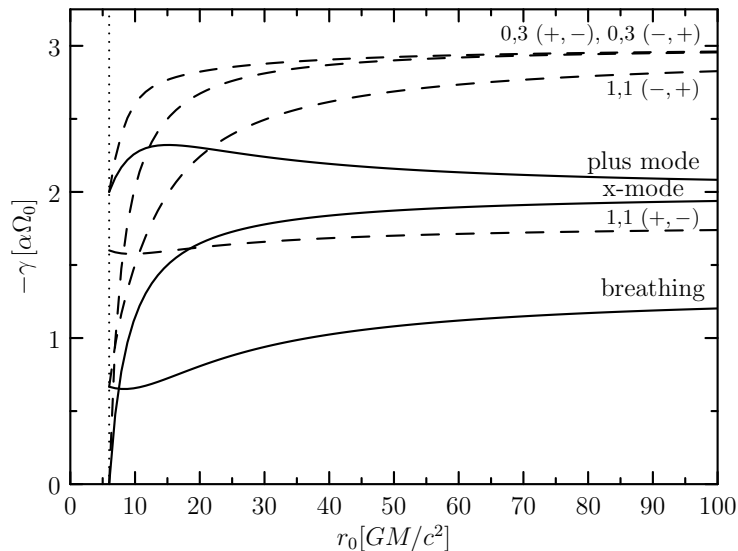
+ Mode



Breathing Mode

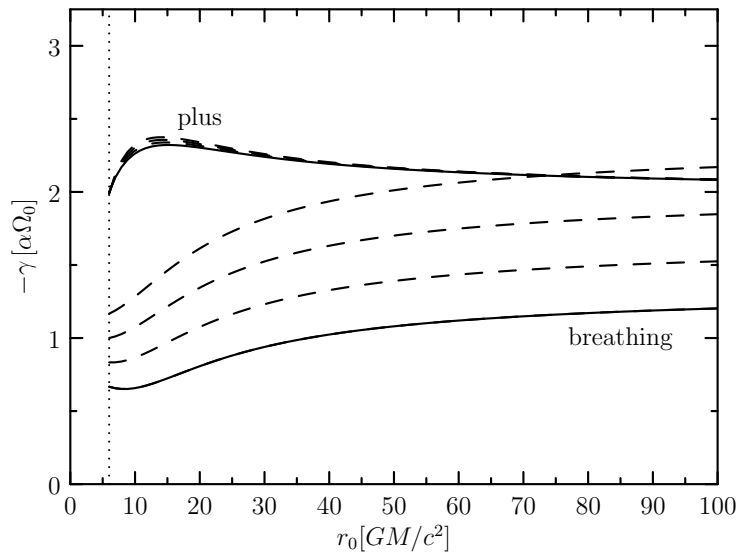


# Radial dependence of the damping rates



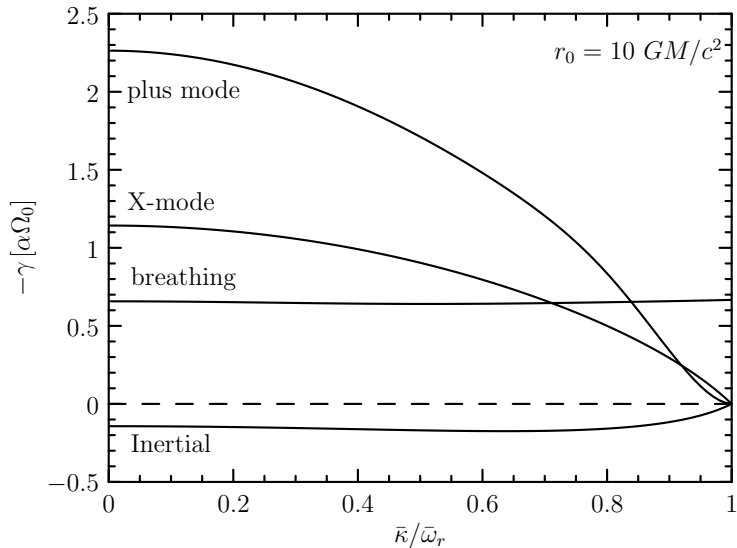
... $\ell = \text{const}$  case

## Dependence on bulk viscosity



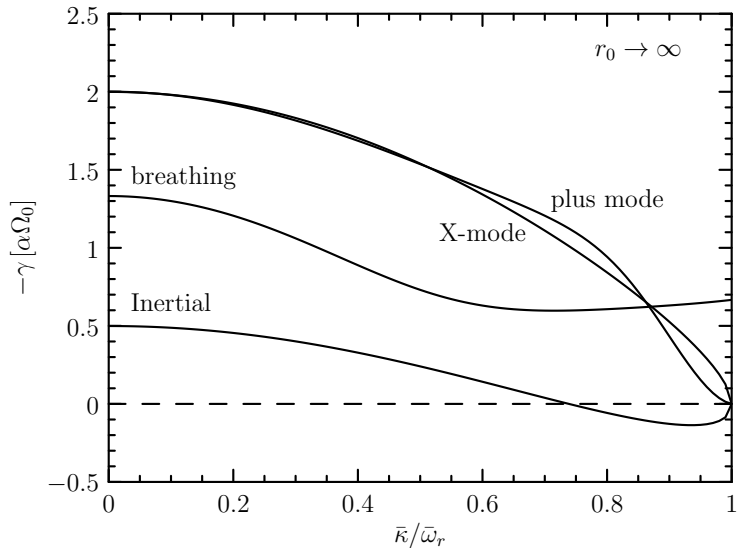
... $\ell = \text{const case}$

## Dependence on the angular momentum profile



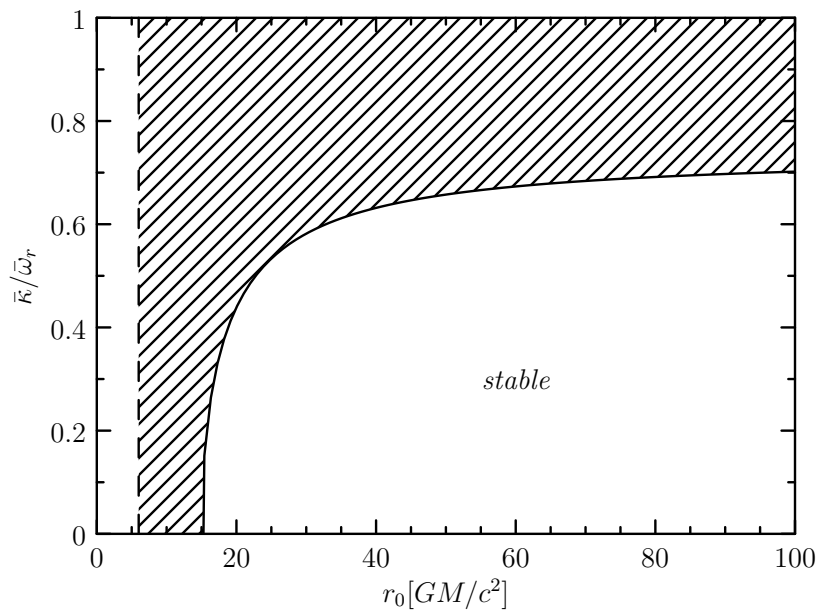
... $r_0 = 10GM/c^2$

## Dependence on the angular momentum profile

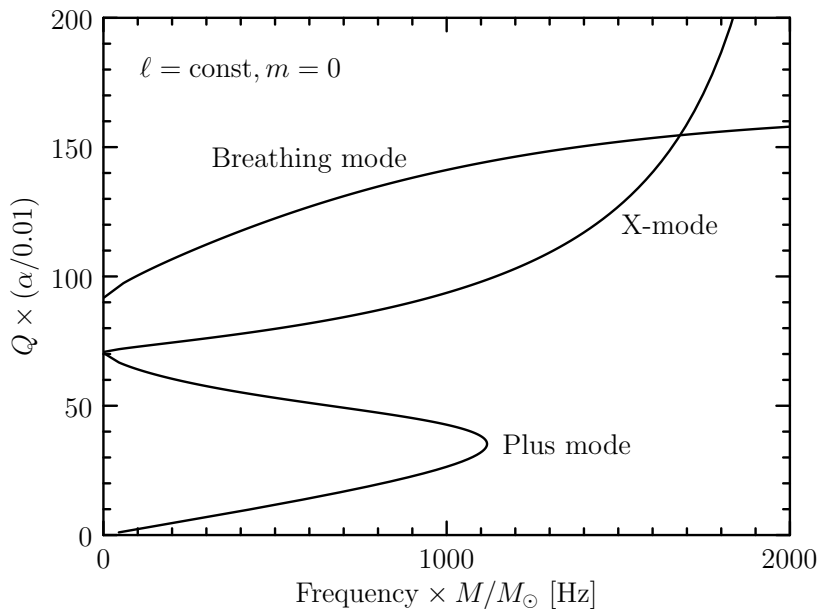


... $r_0 \rightarrow \infty$

## Inertial mode – viscous instability?



## $\nu - Q$ relations



## X-mode and kHz QPO in 4U 1636

